

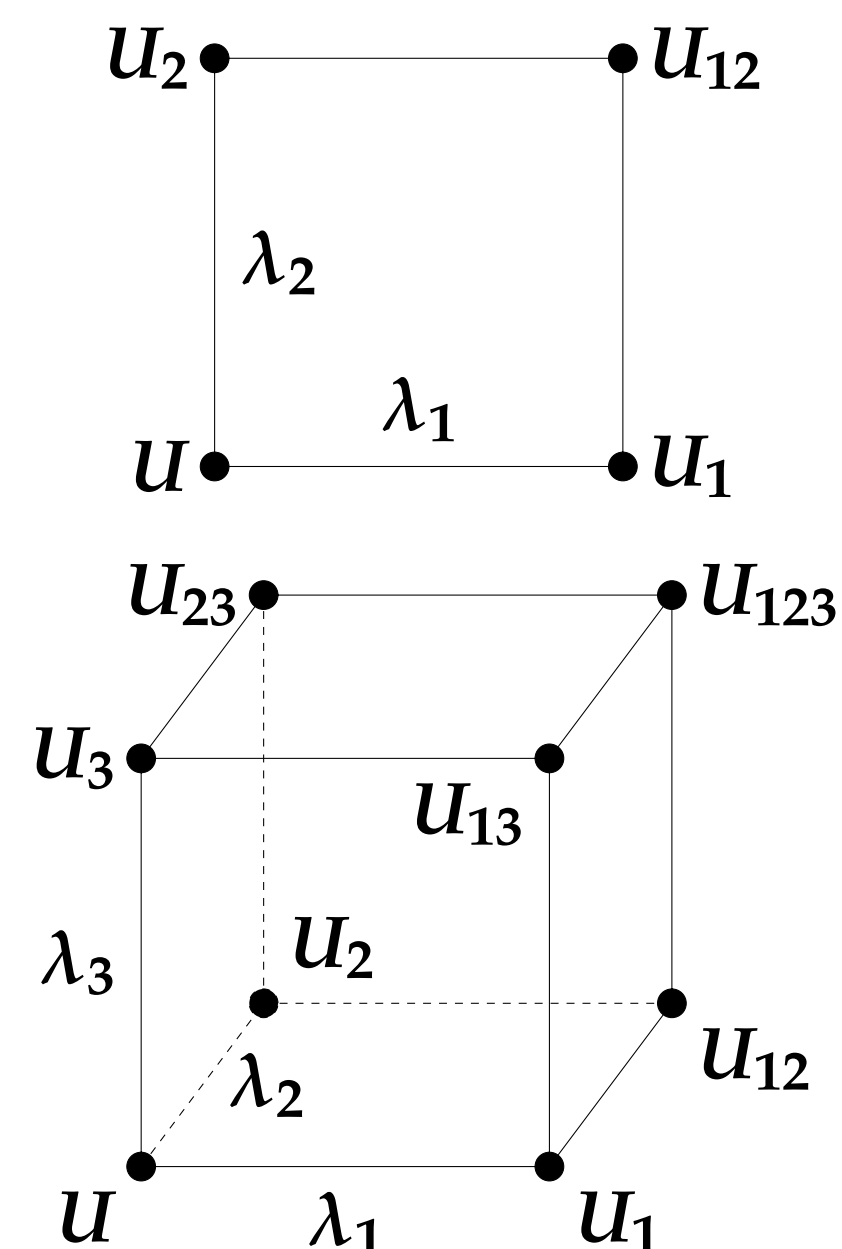
Continuum limits of integrable lattice equations and their variational structure

Multidimensionally consistent lattice equations

Quad equation: $\mathcal{Q}(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2) = 0$,
invariant under symmetries of the square, affine in
each of the entries U, U_1, U_2, U_{12} .

Subscripts of U denote lattice shifts, λ_1 and λ_2 are
lattice parameters.

Integrable if **consistent around the cube**: initial data
 U, U_1, U_2, U_3 lead to a well-defined value of U_{123} .
Then it can be imposed on a higher-dimensional
lattice \mathbb{Z}^N .



Integrable hierarchies of PDEs

Hierarchy of commuting PDEs:

$$\begin{aligned} u_{t_2} &= f_2(u, u_x, u_{xx}, \dots), \\ u_{t_3} &= f_3(u, u_x, u_{xx}, \dots), \dots \end{aligned}$$

A typical example is the potential KdV hierarchy

$$\begin{aligned} u_{t_3} &= 3u_x^2 + u_{xxx}, \\ u_{t_5} &= 10u_x^3 + 5u_{xx}^2 + 10u_x u_{xxx} + u_{xxxxx}, \dots \end{aligned}$$

By commutativity, we can consider a single field
 $u(x, t_2, \dots, t_N)$ on higher-dimensional **multi-time** \mathbb{R}^N .

Lagrangian structure: discrete

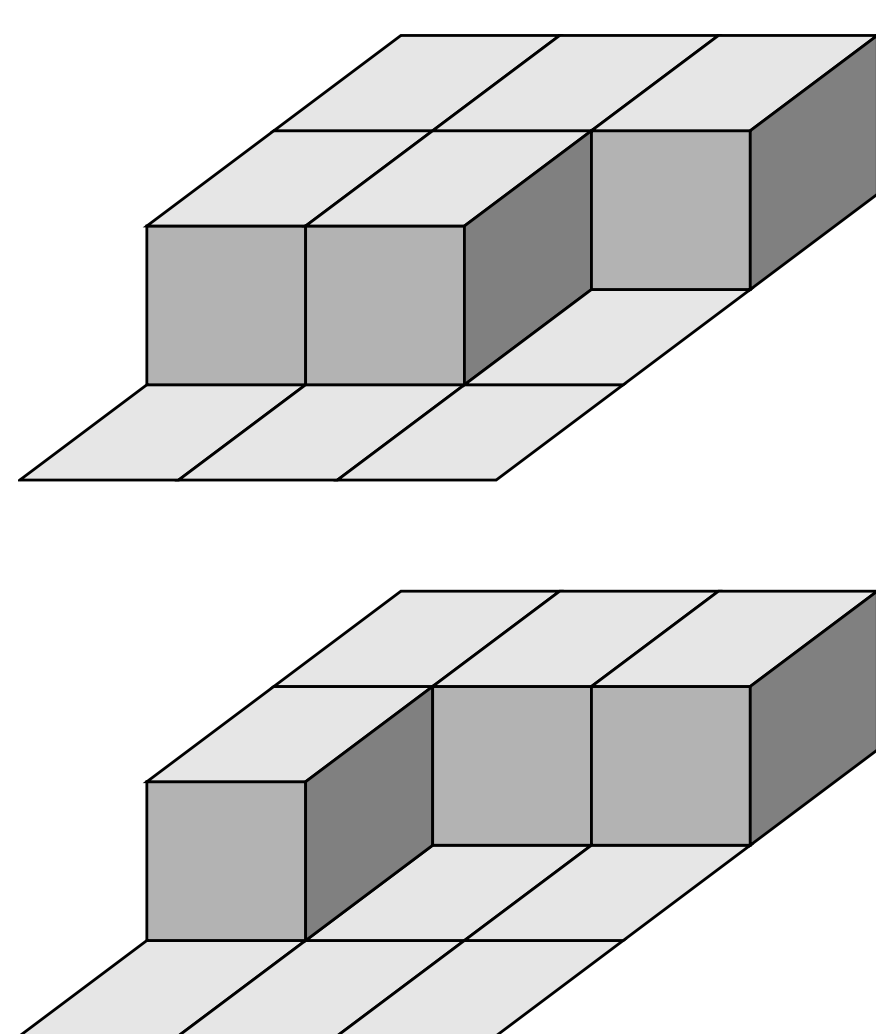
A field $U : \mathbb{Z}^N \rightarrow \mathbb{R}$ is a solution to the
pluri-Lagrangian (or **Lagrangian multiform**)
problem for

$$L(\square_{ij}) = L(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j)$$

if the action

$$\sum_{\square \in \Gamma} L(\square)$$

is critical on all discrete surfaces Γ in \mathbb{Z}^N .



Lagrangian structure: continuous

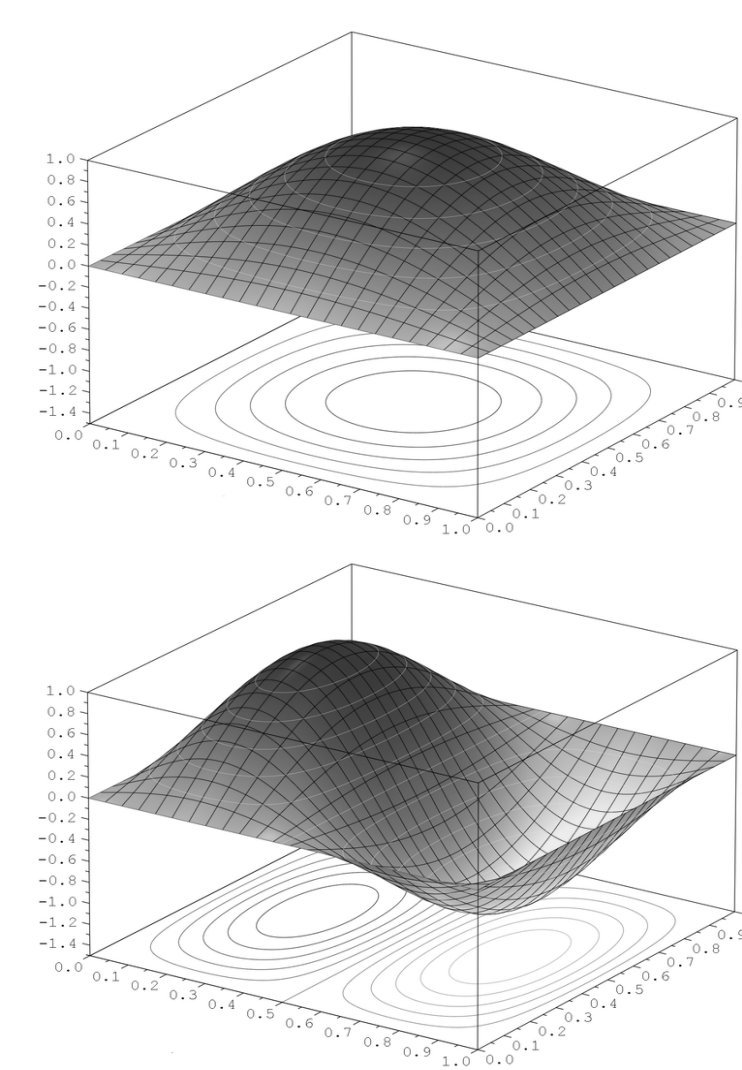
A field $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is a solution to the
pluri-Lagrangian problem for the 2-form

$$\mathcal{L}[u] = \sum_{i,j} \mathcal{L}_{ij}[u] dt_i \wedge dt_j$$

if the action

$$\int_{\Gamma} \mathcal{L}[u]$$

is critical on all smooth surfaces Γ in \mathbb{R}^N .



Embedding the lattice into continuous multi-time: Miwa shifts

Correspondence between discrete and continuous fields:

$$\begin{cases} U = U(\mathbf{n}) &= u(t_1, t_2, \dots, t_N) \\ U_i = U(\mathbf{n} + \mathbf{e}_i) &= u\left(t_1 - 2\lambda_i, t_2 + \lambda_i^2, \dots, t_N + \frac{2(-1)^N \lambda_i^N}{N}\right) \end{cases}$$

[Miwa. On Hirota's difference equations. Proc. Japan Acad. A. 1982]

Skew embedding of the mesh \mathbb{Z}^N into multi-time \mathbb{R}^N .
Discrete U is a sampling of the continuous u .

We write the quad equation in terms of u , expand in λ_1, λ_2 ,
and equate all coefficients of the power series to 0.

↪ Whole hierarchy from single difference equation.

In the leading order, we only see t_1 -derivatives of u , but we want to obtain PDEs. → Leading order cancellation required.

Continuum limit of a Lagrangian

Using **Miwa shifts**, turn discrete $L(\square_{12})$ into power series
 $\mathcal{L}_{\text{disc}}([u], \lambda_1, \lambda_2)$. The action for $\mathcal{L}_{\text{disc}}$ is still a sum.

With the **Euler-Maclaurin formula** we turn the action sum into
an integral, with integrand

$$\mathcal{L}_{\text{Miwa}}([u], \lambda_1, \lambda_2) = \sum_{i,j=0}^{\infty} \frac{B_i B_j}{i! j!} \partial_1^i \partial_2^j \mathcal{L}_{\text{disc}}([u], \lambda_1, \lambda_2),$$

where the differential operators are $\partial_k = \sum_j (-1)^{j+1} \frac{2\lambda_k^j}{j} \frac{d}{dt_j}$.

Then $L(\square_{12}) = \int_{\square_{12}} \mathcal{L}_{\text{Miwa}}([u(t)], \lambda_1, \lambda_2) \eta_1 \wedge \eta_2$,

where η_1 and η_2 are the 1-forms dual to the Miwa shifts.

Pluri-Lagrangian 2-form for the continuum limit hierarchy:

$$\mathcal{L} = \sum_{i < j} \mathcal{L}_{\text{Miwa}}([u], \lambda_i, \lambda_j) \eta_i \wedge \eta_j = \sum_{i < j} \mathcal{L}_{ij}[u] dt_i \wedge dt_j,$$

where $\mathcal{L}_{\text{Miwa}}([u], \lambda_1, \lambda_2) = \sum_{i,j=1}^{\infty} (-1)^{i+j} 4 \frac{\lambda_1^i \lambda_2^j}{i j} \mathcal{L}_{ij}[u]$.

Example

Lattice potential KdV equation:

$$(\lambda_1^{-1} + \lambda_2^{-1} + U_{1,2} - U) (\lambda_2^{-1} - \lambda_1^{-1} + U_2 - U_1) = \lambda_2^{-2} - \lambda_1^{-2} \xrightarrow{\text{Miwa shifts}} \sum_{i,j \geq 0} f_{i,j}[u] \lambda_1^i \lambda_2^j = 0 \xrightarrow{f_{i,j}=0}$$

Lagrangian:

$$L = \frac{1}{2} (U - U_{i,j} - \lambda_i^{-1} - \lambda_j^{-1}) (U_i - U_j + \lambda_i^{-1} - \lambda_j^{-1}) + (\lambda_i^{-2} - \lambda_j^{-2}) \log \left(1 + \frac{U_i - U_j}{\lambda_i^{-1} - \lambda_j^{-1}} \right).$$

$\xrightarrow{\text{Miwa shifts, EM formula}}$

$$\begin{cases} \mathcal{L}_{1,3} = -2u_1^3 - u_1 u_{111} + u_1 u_3 \\ \mathcal{L}_{1,5} = -5u_1^4 + 10u_1 u_{111}^2 - u_{111}^2 + u_1 u_5 \\ \mathcal{L}_{3,5} = 6u_1^5 - 15u_1^2 u_{111}^2 + 20u_1^3 u_{111} - 10u_1^3 u_3 + 7u_{111}^2 u_{111} + 6u_1 u_{111}^2 - 12u_1 u_{111} u_{1111} \\ \quad + 3u_1^2 u_{11111} + 20u_1 u_{111} u_{113} - 5u_{111}^2 u_3 - 10u_1 u_{111} u_3 + 3u_1^2 u_5 - u_{1111}^2 \\ \quad + u_{111} u_{11111} - 2u_{111} u_{113} + u_1 u_{115} + 2u_{1111} u_{113} - u_{11} u_{15} - u_{11111} u_3 + u_{111} u_5 \end{cases} \quad \text{where } u_i = u_{t_i}.$$