A Lagrangian perspective on integrability

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1. Overview

Most nonlinear differential equations produce dynamics that are very hard to understand and impossible to solve explicitly. An exception are *integrable systems*, which are tamed by some underlying structure. Often, this structure consists of a number of symmetries. Hence:

An equation is integrable if has sufficiently many symmetries.

Each symmetry, in it infinitesimal form, defines a differential equation. These can be taken together with the original equation to form a system of differential equations. Hence an alternative characterisation of integrability is:

An equation is integrable if it is part of a sufficiently large family of compatible equations.

A weak but useful interpretation of "compatible" is that the flows of the different equations commute with each other. More sophisticated interpretations are traditionally given in terms of Hamiltonian mechanics (see boxes 2 and 3). Below we present a Lagrangian perspective on integrability.

2. Lagrangian/Hamiltonian mechanics

Lagrangian mechanics

$$\delta \int L(q(t),\dot q(t))\,\mathrm{d}t = 0 \quad \Leftrightarrow \quad \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q} = 0.$$

This is equivalent, through

$$p = \frac{\partial L}{\partial \dot q}, \qquad H = \dot q p - L,$$

to Hamiltonian mechanics:

$$\dot q = \frac{\partial H(q,p)}{\partial p}, \qquad \dot p = -\frac{\partial H(q,p)}{\partial q}.$$

In terms of the the Poisson bracket

$$\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p},$$

Hamiltonian dynamics are given by

$$\frac{\mathrm{d}f(q,p)}{\mathrm{d}t} = \{H,f\}.$$

Note that $\{,\}$ is skew-symmetric, and f is an integral of motion if and only if $\{H,f\} = 0$.

3. Commuting flows and integrability

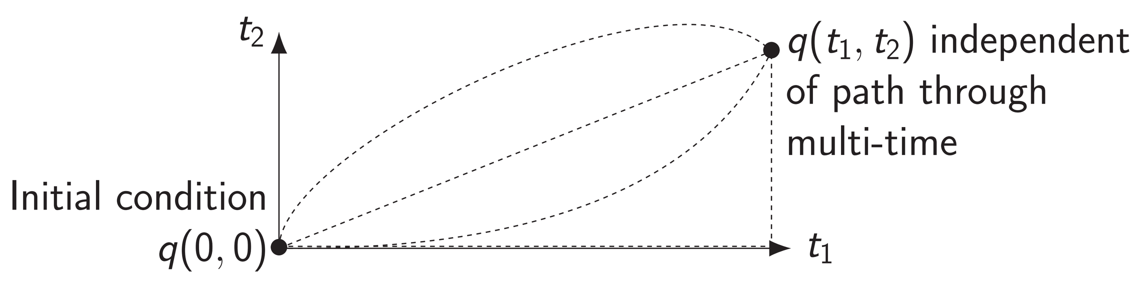
A Hamiltonian system with n degrees of freedom is *Liouville integrable* if its Hamiltonian H is part of a family $H_1,\dots,H_n$ of functionally independent Hamiltonians that satisfy $\{H_i,H_j\} = 0$.

- Each H_i defines a dynamical system for which all H_j are integrals of motion.
- The flows of these systems commute with each other, so we can consider simultaneous solutions $q(t_1,\dots,t_n)$ of $\frac{\mathrm{d}f}{\mathrm{d}t_i} = \{H_i,f\}$, $i = 1,\dots,n$.
- Dynamics are constrained to a common level set of the H_i. If this level set is compact, it is a topological torus and trajectories are (quasi-)periodic.

Liouville integrability defines what "compatible equations" are for Hamiltonian systems. *Can we characterise this using a variational principle?*

4. Variational principle in multi-time

Minimal example: consider two commuting Lagrangian ODEs.



(Additional commuting equations can be accommodated by increasing the dimension n of multi-time: $q : \mathbb{R}^n \to \mathbb{R}$.)

Classical Lagrangian description: the actions

$$\int L_1(q,q_{t_1})\,\mathrm{d}t_1 \quad \text{and} \quad \int L_2(q,q_{t_1},q_{t_2})\,\mathrm{d}t_2$$

are extremised by q. (Subscripts denote partial derivatives.)

Pluri-Lagrangian principle / Lagrangian multiform principle:

Consider the 1-form $\mathcal{L}[q] = L_1(q,q_{t_1})\,\mathrm{d}t_1 + L_2(q,q_{t_1},q_{t_2})\,\mathrm{d}t_2$.

For every curve $\gamma : [0,1] \to \mathbb{R}^2$ in multi-time, $q : \mathbb{R}^2 \to \mathbb{R}$ extremises $\int_\gamma \mathcal{L}[q]$.

5. Multi-time Euler-Lagrange equations

Taking variations of q leads to the the usual Euler-Lagrange equations

$$\frac{\delta_i L_i}{\delta q} := \frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial L_i}{\partial q_{t_i}} + \ldots = 0 \qquad (1)$$

as well as

$$\frac{\delta_1 L_1}{\delta q_{t_1}} = \frac{\delta_2 L_2}{\delta q_{t_2}} \quad \Leftrightarrow \quad \frac{\partial L_1}{\partial q_{t_1}} + \ldots = \frac{\partial L_2}{\partial q_{t_2}} + \ldots, \qquad (2)$$

$$\frac{\delta_i L_i}{\delta q_{t_j}} = 0 \quad \Leftrightarrow \quad \frac{\partial L_i}{\partial q_{t_j}} + \ldots = 0 \qquad \text{if } i \neq j. \qquad (3)$$

Theorem. Equations (1)–(3) characterise extremisers q for the variational principle in multi-time [7, 8].

Theorem. The following are equivalent:

- $\mathrm{d}\mathcal{L} = 0$ when evaluated on solutions q to the multi-time Euler-Lagrange equations (1)–(3).
- The action is critical with respect to variations of the curve γ (as opposed to variations of q only).
- The corresponding Hamiltonians are in involution [10].

6. Example

The Kepler Lagrangian

$$L_1 = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

can be combined with

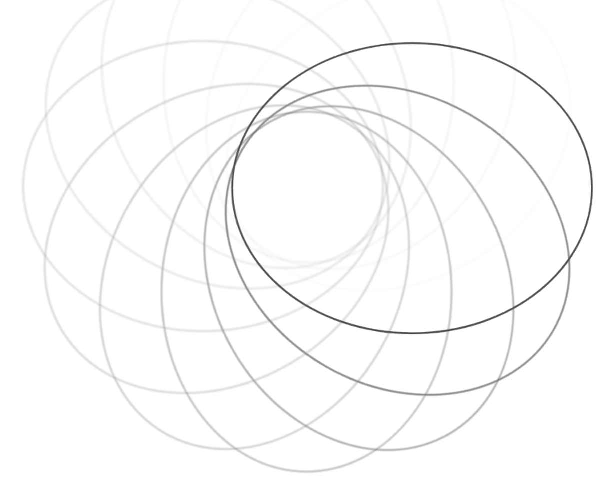
$$L_2 = q_{t_1}\cdot q_{t_2} + (q_{t_1} \times q)\cdot e,$$

yielding multi-time EL equations

$$q_{t_1 t_1} = -\frac{q}{|q|^3}, \qquad \text{(inverse square law)}$$

$$q_{t_2} = e \times q \qquad \text{(rotation)}$$

and consequences of these.



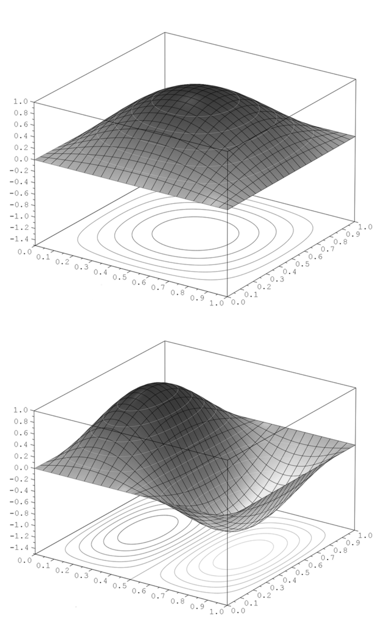
7. Partial differential equations

PDEs will typically share their spatial coordinates but have separate time coordinates.

For 1+1 dimensional PDEs, a pluri-Lagrangian structure is a 2-form

$$\mathcal{L} = \sum_{i,j} L_{ij}[u]\,\mathrm{d}t_i \wedge \mathrm{d}t_j$$

whose integral must be extremised over every surface in multi-time. We identify $t_1 = x$.



The multi-time Euler-Lagrange equations are

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

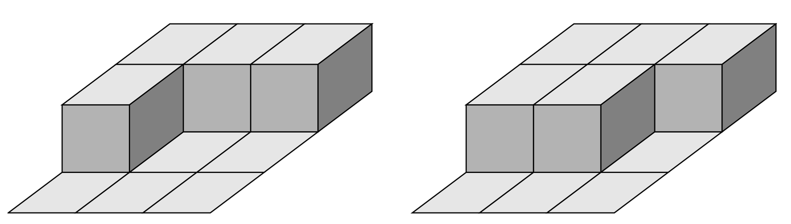
$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{It_k t_i}} = 0 \qquad \forall I.$$

Lagrangian 2-forms are known for potential KdV [8], Sine-Gordon and modified KdV [4], AKNS [5],... A 3-form is known for KP [6].

8. Difference equations

For difference equations on elementary squares of $\mathbb{Z}^2$, the action should be extremised on any discrete surface in $\mathbb{Z}^3$.



This perspective was important to investigate integrability for partial difference equations [3, 1, 2]. Continuum limits were studied in [9].

Example: the lattice potential KdV equation

$$(U_{i,j} - U_{i+1,j+1})(U_{i+1,j} - U_{i,j+1}) = \alpha_1 - \alpha_2,$$

where $U : \mathbb{Z}^2 \to \mathbb{R} : (i,j) \mapsto U_{i,j}$ and α_1, α_2 are lattice parameters, has Lagrangian

$$L = (U_{i+1,j} - U_{i,j+1})U_{i,j} - (\alpha_1 - \alpha_2)\ln(U_{i+1,j} - U_{i,j+1}).$$

It can be imposed on each elementary square in $\mathbb{Z}^N$ by adding lattice parameters $\alpha_3,\dots,\alpha_N$.

9. Outlook

The theory of pluri-Lagrangian (Lagrangian multiform) systems shows that a variational description of integrable systems is feasible. In certain contexts, like discretisation or quantisation, the Lagrangian point of view may well be preferable.

Some topics of ongoing research are:

- Lagrangian k-forms for *symmetries that do not commute with each other*, by replacing euclidean multi-time with a Lie group. In particular, this is relevant to *super-integrable systems*
- Applying the formalism to *non-integrable systems* with certain symmetries.
- Extending the theory to systems with *infinite-dimensional symmetry groups* (as in Noether's second theorem), in particular to *gauge theories* of physics.

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