A variational structure for integrable hierarchies

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Integrable systems

Many different contexts, many different definitions.

Intuitive meaning

An integrable system is (a system of) nonlinear differential or difference equation(s), that behaves as if it were linear:

- Solvability (in some sense)
- Superposition principle (for special solutions)
- Rich hidden structure explaining nice behavior

That structure can take many shapes. For us, it will always be a variation the following idea:

Vague definition

An equation is integrable if it is part of a "sufficiently large" system of "compatible" equations.

Hamiltonian Systems

Hamilton function

$$H : \mathbb{R}^{2N} \cong T^*Q \to \mathbb{R} : (q, p) \mapsto H(q, p)$$

determines dynamics:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Geometric interpretation:

- Phase space T^*Q with canonical symplectic 2-form ω
- flow along vector field X_H determined by $\iota_{X_H}\omega = \mathrm{d}H$
- the flows consists of symplectic maps and preserves H.

Poisson Brackets

Poisson bracket of two functionals on T^*Q:

$$\{f,g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \quad = \omega(\nabla f, \nabla g)$$

Dynamics of a Hamiltonian system:

$$\dot{q}_i = \{q_i, H\}, \qquad \dot{p}_i = \{p_i, H\}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} f(q,p) = \{f(q,p), H\}$$

Properties:

$$\begin{aligned} \text{anti-symmetry: } & \{f,g\} = -\{g,f\} \\ \text{bilinearity: } & \{f, g + \lambda h\} = \{f,g\} + \lambda \{f,h\} \\ \text{Leibniz property: } & \{f, gh\} = \{f,g\}h + g\{f,h\} \\ \text{Jacobi identity: } & \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \end{aligned}$$

Any bracket with these properties supports Hamiltonian systems.

Liouville-Arnold integrability

A Hamiltonian system with Hamilton function $H : \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville-Arnold integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \ldots H_N$ such that

$$\{H_i, H_j\} = 0.$$

In particular, this implies that the flows commute.
(In fact $\{H_i, H_j\} = const$ would be sufficient.)

The evolution of a Liouville-Arnold integrable system is linear on a topological N-torus.

Lagrangian Mechanics

Lagrange function

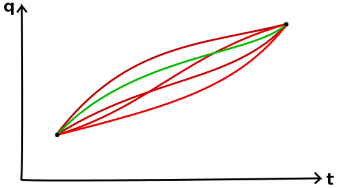
$$L : \mathbb{R}^{2N} \cong TQ \to \mathbb{R} : (q, \dot{q}) \mapsto L(q, \dot{q})$$

determines dynamics:

Flow along curves $q(t)$ that minimize (or are critical points of) the action

$$\int_{t_0}^{t_1} L(q(t), \dot{q}(t)) \, \mathrm{d}t$$

where the integration interval $[t_0, t_1]$ and the boundary values $q(t_0)$ and $q(t_1)$ are fixed.



Lagrangian Mechanics

This is called a variational principle, because the equations of motion can be derived formally using infinitesimal variations δq of the curve q. The curve is a critical points if

$$
\begin{aligned}
0 &= \delta \int_{t_0}^{t_1} L(q(t), \dot{q}(t)) \, \mathrm{d}t \\
&= \int_{t_0}^{t_1} \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \, \mathrm{d}t \\
&= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \right) \delta q \, \mathrm{d}t + \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \bigg|_{t_0}^{t_1}
\end{aligned}
$$

Euler-Lagrange Equation(s):

$$
\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} = 0 \qquad \text{for scalar } q
$$

$$
\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} = 0 \quad \text{for } i = 1, \ldots, N \qquad \text{if } q \in \mathbb{R}^N
$$

Lagrangian Mechanics

This is called a variational principle, because the equations of motion can be derived formally using infinitesimal variations δq of the curve q. The curve is a critical points if

$$\begin{aligned} 0 &= \delta \int_{t_0}^{t_1} L(q(t), \dot{q}(t)) \,\mathrm{d}t \\ &= \int_{t_0}^{t_1} \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \,\mathrm{d}t \\ &= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \right) \delta q \,\mathrm{d}t + \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \Big|_{t_0}^{t_1} \end{aligned}$$

Euler-Lagrange Equation(s):

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} = 0 \qquad \text{for scalar } q$$

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} = 0 \quad \text{for } i = 1, \ldots, N \qquad \text{if } q \in \mathbb{R}^N$$

Legendre transformation

Relates Hamiltonian and Lagrangian formalism:

$$p\dot{q} = H(q,p) + L(q,\dot{q}).$$

Differentiating w.r.t. $\dot{q}$, p and q,

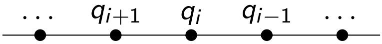
$$\begin{aligned}
p &= \frac{\partial L}{\partial \dot{q}} \\
\dot{q} &= \frac{\partial H}{\partial p} \\
0 &= \frac{\partial H}{\partial q} + \frac{\partial L}{\partial q} \quad = \left(\frac{\partial H}{\partial q} + \dot{p} \right) + \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \right),
\end{aligned}$$

establishes equivalence between Hamiltonian and Lagrangian equations of motion.

For Hamiltonian systems “built on” different Poisson brackets the relation is not so obvious.

Example: Toda Lattice

Configuration variable $q \in \mathbb{R}^N$, positions of N particles on a line.



Lagrangian:

$$L(q, \dot{q}) = \sum_{i=1}^{N} \left(\frac{1}{2} \dot{q}_i^2 - e^{q_{i+1} - q_i} \right),$$

The Euler-Lagrange equations are

$$0 = \frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} = e^{q_i - q_{i-1}} - e^{q_{i+1} - q_i} - \ddot{q}_i,$$

so the dynamics are determined by

$$\ddot{q}_i = e^{q_i - q_{i-1}} - e^{q_{i+1} - q_i}$$

This is the first of an infinite hierarchy of compatible ODEs.

Lagrangian PDEs

Lagrangian density $L(v, v_t, v_x, v_{tt}, v_{xt}, v_{xx}, \ldots)$

Action $\mathcal{S} = \int L \,\mathrm{d}x\,\mathrm{d}t$

Look for a function v that is a critical point of the action, i.e. for arbitrary infinitesimal variations δv:

$$0 = \delta\mathcal{S} = \int \delta L \,\mathrm{d}x\,\mathrm{d}t = \int \sum_I \frac{\partial L}{\partial v_I}\delta v_I \,\mathrm{d}x\,\mathrm{d}t$$

$$= \int \sum_I (-1)^{|I|} \left(\mathrm{D}_I \frac{\partial L}{\partial v_I}\right) \delta v \,\mathrm{d}x\,\mathrm{d}t$$

Euler-Lagrange equation:

$$\frac{\delta L}{\delta v} := \sum_I (-1)^{|I|} \mathrm{D}_I \frac{\partial L}{\partial v_I} = 0$$

If $I = (i_1, \ldots, i_k)$ then $D_I = \frac{\mathrm{d}^{i_1}}{\mathrm{d}^{i_1} t_1} \cdots \frac{\mathrm{d}^{i_k}}{\mathrm{d}^{i_k} t_k}$ and $v_I = D_I v$.

Example: KdV equation

Lagrangian density $L = \frac{1}{2} v_x v_t - v_x^3 - \frac{1}{2} v_x v_{xxx}$

Euler-Lagrange Equation:

$$
\begin{aligned}
0 = \frac{\delta L}{\delta v} &= \sum_I (-1)^{|I|} \mathrm{D}_I \frac{\partial L}{\partial v_I} \\
&= \frac{\partial L}{\partial v} - \mathrm{D}_t \frac{\partial L}{\partial v_t} - \mathrm{D}_x \frac{\partial L}{\partial v_x} + \mathrm{D}_{xx} \frac{\partial L}{\partial v_{xx}} - \mathrm{D}_{xxx} \frac{\partial L}{\partial v_{xxx}} + \dots \\
&= -\frac{1}{2} \mathrm{D}_t(v_x) - \frac{1}{2} \mathrm{D}_x(v_t) + 3\mathrm{D}_x(v_x^2) + \frac{1}{2} \mathrm{D}_x(v_{xxx}) + \frac{1}{2} \mathrm{D}_{xxx}(v_x) \\
&= -v_{xt} + 6 v_x v_{xx} + v_{xxxx}
\end{aligned}
$$

$$\Rightarrow v_{xt} = 6 v_x v_{xx} + v_{xxxx}$$

Substitute $u = v_x$ to find the Korteweg-de Vries equation

$$u_t = 6 u u_x + u_{xxx}.$$

Or integrate to find the Potential Korteweg-de Vries equation

$$v_t = 3 v_x^2 + v_{xxx}.$$

Main question

Integrable systems like the Toda lattice and the KdV equation come with infinite hierarchies. Each individual equation is Lagrangian/Hamiltonian.

On the Hamiltonian side it is clear when the equations of a hierarchy fit together: $\{H_i, H_j\} = 0$.

What about the Lagrangian side?

Is there a variational description of an integrable hierarchy?

1. Background
   - Integrable systems
   - Hamiltonian systems
   - Variational systems

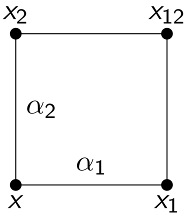
2. Discrete pluri-Lagrangian systems

3. Continuous pluri-Lagrangian systems
   - Definition and characterization
   - Examples: Toda and KdV
   - Relation to other notions

Inspiration: lattice equations

Quad equation:

$$Q(x, x_1, x_2, x_{12}, \alpha_1, \alpha_2) = 0$$

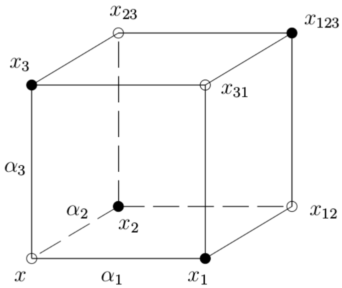


Subscripts of x denote lattice shifts, α_1, α_2 are parameters.
Invariant under symmetries of the square, affine in each of x, x_1, x_2, x_{12}.

Integrability for systems quad equations:
Multi-dimensional consistency of

$$Q(x, x_i, x_j, x_{ij}, \alpha_i, \alpha_j) = 0,$$

i.e. the three ways of calculating x_{123} give the same result.

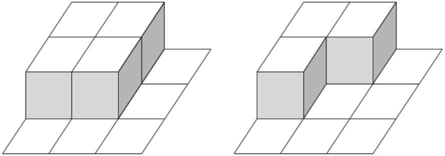


Inspiration: lattice equations

- Classification integrable quad equations:
  [Adler, Bobenko, Suris. Classification of integrable equations on quad-graphs. The consistency approach. Commun. Math. Phys. 2003.]
- Variational formulation for all of them:
  [Lobb, Nijhoff. Lagrangian multiforms and multidimensional consistency. J. Phys. A. 2009.]

Pluri-Lagrangian structure for quad equations

For some discrete 2-form $\mathcal{L}(\sigma_{ij}) = \mathcal{L}(x, x_i, x_j, x_{ij}, \alpha_i, \alpha_j)$, the action $\sum_{\sigma_{ij} \in S} \mathcal{L}(\sigma_{ij})$ is critical on all 2-surfaces S in $\mathbb{N}^N$ simultaneously.



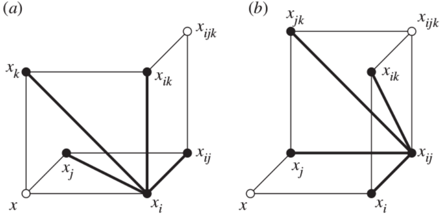
Furthermore, the value of the action does not depend on the surface, i.e. the discrete 2-form $\mathcal{L}$ is closed.

Example

For the discrete KdV equation $(x - x_{ij})(x_i - x_j) - \alpha_i + \alpha_j = 0$ we have the Lagrangian

$$\mathcal{L}(x, x_i, x_j, x_{ij}, \alpha_i, \alpha_j) = (x_i - x_j)x - (\alpha_i - \alpha_i)\log(x_i - x_j)$$

We look at elementary corners of an arbitrary surface:



Depending on the orientation, we get the criticality conditions

$$(a) \quad x_{ij} - x_{ik} - \frac{\alpha_i - \alpha_k}{x_i - x_k} + \frac{\alpha_i - \alpha_j}{x_i - x_j} = 0,$$

$$(b) \quad x_i - x_j - \frac{\alpha_j - \alpha_k}{x_{ij} - x_{ik}} + \frac{\alpha_i - \alpha_k}{x_{ij} - x_{jk}} = 0$$

Example

$$(a) \quad x_{ij} - x_{ik} - \frac{\alpha_i - \alpha_k}{x_i - x_k} + \frac{\alpha_i - \alpha_j}{x_i - x_j} = 0,$$

$$(b) \quad x_i - x_j - \frac{\alpha_j - \alpha_k}{x_{ij} - x_{ik}} + \frac{\alpha_i - \alpha_k}{x_{ij} - x_{jk}} = 0$$

The conditions for other elementary shapes (flat or edge), follow from these.

They are consequences of, but not equivalent to, the discrete KdV equation

$$(x - x_{ij})(x_i - x_j) - \alpha_i + \alpha_j = 0.$$

Recall KdV: The Euler-Lagrange equation was a consequence of, but not equivalent to, the potential KdV equation.

Details e.g. in [Boll, Petrera, Suris. What is integrability of discrete variational systems? Proc. R. Soc. A. 2014.]

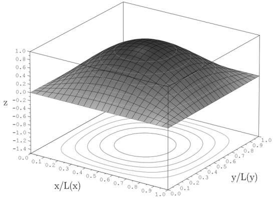
Continuous analogue

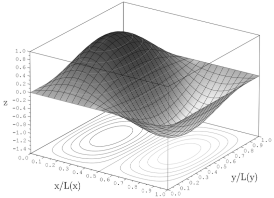
2D Pluri-Lagrangian structure

Field u on multi-time $\mathbb{R}^N$,

$$\mathcal{L} = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j.$$

Action $\int_S \mathcal{L}$ is critical on all smooth 2-surfaces S in $\mathbb{R}^N$.





Possible for any dimension: field u on multi-time $\mathbb{R}^N$,

$$\mathcal{L} = \sum_{i_1,\ldots,i_d} L_{i_1 \ldots i_d}[u] \, \mathrm{d}t_{i_1} \wedge \ldots \wedge \mathrm{d}t_{i_d}.$$

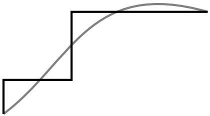
Action $\int_S \mathcal{L}$ is critical on all smooth d-surfaces S in $\mathbb{R}^N$.

The 1D case

Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i[u] \,\mathrm{d}t_i$

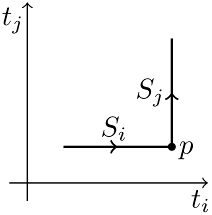
Lemma

If the action $\int_S \mathcal{L}$ is critical on all stepped curves S in $\mathbb{R}^N$, then it is critical on all smooth curves.



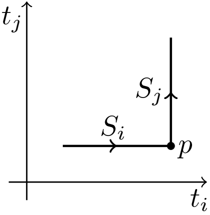
Indeed, $\int_{S_1} \mathcal{L} - \int_{S_2} \mathcal{L} = \int_M \mathrm{d}\mathcal{L}$, where $\partial M = S_1 \cup S_2$. By choosing a fine approximation, M can be made arbitrarily small.

Variations are local, so it is sufficient to look at a general L-shaped curve $S = S_i \cup S_j$.



Multi-time Euler-Lagrange equations for the 1D case

The variation of the action on S_i is



$$\delta \int_{S_i} L_i \,\mathrm{d}t_i = \int_{S_i} \sum_I \frac{\partial L_i}{\partial u_I} \delta u_I \,\mathrm{d}t_i$$

$$= \int_{S_i} \sum_{I \not\ni t_i} \frac{\delta_i L_i}{\delta u_I} \delta u_I \,\mathrm{d}t_i + \sum_I \frac{\delta_i L_i}{\delta u_{It_i}} \delta u_I \bigg|_p ,$$

where I denotes a multi-index, and

$$\frac{\delta_i L_i}{\delta u_I} = \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial L_i}{\partial u_{It_i^\alpha}} = \frac{\partial L_i}{\partial u_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial u_{It_i}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial L_i}{\partial u_{It_i^2}} - \ldots$$

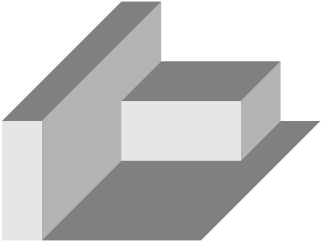
Multi-time Euler-Lagrange equations for curves, $\mathcal{L} = \sum_i L_i[u] \,\mathrm{d}t_i$

$$\frac{\delta_i L_i}{\delta u_I} = 0 \qquad \forall I \not\ni t_i \qquad \text{and} \qquad \frac{\delta_i L_i}{\delta u_{It_i}} = \frac{\delta_j L_j}{\delta u_{It_j}} \qquad \forall I,$$

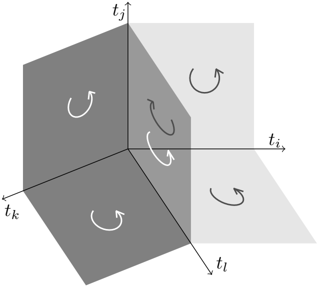
The 2D case

Consider a Lagrangian two-form

$$\mathcal{L} = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j.$$



It is sufficient to look at stepped surfaces and their elementary corners.



An arbitrary number k of planes can meet in one point, forming a k-flower.
A k-flower can be decomposed into 3-flowers.

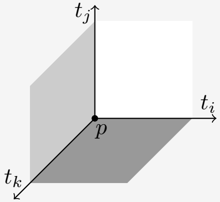
The 2D case

Multi-time EL equations for 2D surfaces, $\mathcal{L} = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{It_k t_i}} = 0 \qquad \forall I.$$



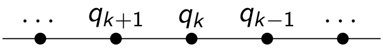
Where

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial u_{I t_i^\alpha t_j^\beta}}$$

$$= \frac{\partial L_{ij}}{\partial u_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_{ij}}{\partial u_{It_i}} - \frac{\mathrm{d}}{\mathrm{d}t_j} \frac{\partial L_{ij}}{\partial u_{It_j}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial L_{ij}}{\partial u_{It_i t_i}} + \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\mathrm{d}}{\mathrm{d}t_j} \frac{\partial L_{ij}}{\partial u_{It_i t_j}} + \frac{\mathrm{d}^2}{\mathrm{d}t_j^2} \frac{\partial L_{ij}}{\partial u_{It_j t_j}}$$

Example: Toda hierarchy

$$\mathcal{L} = L_1[q]\mathrm{d}t_1 + L_2[q]\mathrm{d}t_2$$



$$L_1[q] = \sum_k \frac{1}{2}(q_k)_{t_1}^2 - e^{q_k - q_{k-1}}$$

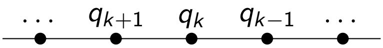
$$L_2[q] = \sum_k (q_k)_{t_1}(q_k)_{t_2} + \frac{1}{3}(q_k)_{t_1}^3 + ((q_{k-1})_{t_1} + (q_k)_{t_1})e^{q_k - q_{k-1}}$$

$$\frac{\delta_1 L_1}{\delta q_k} = 0 \quad \Rightarrow \quad (q_k)_{t_1 t_1} = e^{q_{k+1} - q_k} - e^{q_k - q_{k-1}}$$

$$\frac{\delta_2 L_2}{\delta q_k} = 0 \quad \Rightarrow \quad (q_k)_{t_1 t_2} = ((q_k)_{t_1} + (q_{k+1})_{t_1})e^{q_{k+1} - q_k} - ((q_{k-1})_{t_1} + (q_k)_{t_1})e^{q_k - q_{k-1}}$$

Example: Toda hierarchy

$\mathcal{L} = L_1[q]\mathrm{d}t_1 + L_2[q]\mathrm{d}t_2$



$$L_1[q] = \sum_k \frac{1}{2}(q_k)_{t_1}^2 - e^{q_k - q_{k-1}}$$

$$L_2[q] = \sum_k (q_k)_{t_1}(q_k)_{t_2} + \frac{1}{3}(q_k)_{t_1}^3 + ((q_{k-1})_{t_1} + (q_k)_{t_1})e^{q_k - q_{k-1}}$$

$$\frac{\delta_1 L_1}{\delta q_k} = 0 \quad \Rightarrow \quad (q_k)_{t_1 t_1} = e^{q_{k+1} - q_k} - e^{q_k - q_{k-1}}$$

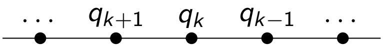
$$\frac{\delta_2 L_2}{\delta q_k} = 0 \quad \Rightarrow \quad (q_k)_{t_1 t_2} = ((q_k)_{t_1} + (q_{k+1})_{t_1})e^{q_{k+1} - q_k} - ((q_{k-1})_{t_1} + (q_k)_{t_1})e^{q_k - q_{k-1}}$$

$$\frac{\delta_2 L_2}{\delta (q_k)_{t_1}} = 0 \quad \Rightarrow \quad (q_k)_{t_2} = -(q_k)_{t_1}^2 + e^{q_{k+1} - q_k} - e^{q_k - q_{k-1}}$$

$$\frac{\delta_1 L_1}{\delta (q_k)_{t_1}} = \frac{\delta_2 L_2}{\delta (q_k)_{t_2}} \quad \Rightarrow \quad (q_k)_{t_1} = (q_k)_{t_1}$$

Example: Toda hierarchy

$$\mathcal{L} = L_1[q]\mathrm{d}t_1 + L_2[q]\mathrm{d}t_2$$



$$L_1[q] = \sum_k \frac{1}{2}(q_k)_{t_1}^2 - e^{q_k - q_{k-1}}$$

$$L_2[q] = \sum_k (q_k)_{t_1}(q_k)_{t_2} + \frac{1}{3}(q_k)_{t_1}^3 + ((q_{k-1})_{t_1} + (q_k)_{t_1})e^{q_k - q_{k-1}}$$

$$\frac{\delta_1 L_1}{\delta q_k} = 0 \quad \Rightarrow \quad (q_k)_{t_1 t_1} = e^{q_{k+1}-q_k} - e^{q_k - q_{k-1}} \qquad (1)$$

$$\frac{\delta_2 L_2}{\delta q_k} = 0 \quad \Rightarrow \quad (q_k)_{t_1 t_2} = ((q_k)_{t_1} + (q_{k+1})_{t_1})e^{q_{k+1}-q_k} - ((q_{k-1})_{t_1} + (q_k)_{t_1})e^{q_k - q_{k-1}}$$

(follows from (1) and (2))

$$\frac{\delta_2 L_2}{\delta (q_k)_{t_1}} = 0 \quad \Rightarrow \quad (q_k)_{t_2} = -(q_k)_{t_1}^2 + e^{q_{k+1}-q_k} - e^{q_k - q_{k-1}} \qquad (2)$$

$$\frac{\delta_1 L_1}{\delta(\dots)} = \frac{\delta_2 L_2}{\delta(\dots)} \quad \Rightarrow \quad (q_k)_{t_1} = (q_k)_{t_1}$$

(trivial)

Example: Toda hierarchy

- The pluri-Lagrangian formalism produces the 2nd flow in its evolutionary form: $(q_k)_{t_2} = \dots$. (Same for higher flows.)

  The classical Lagrangian formalism only gives the differentiated flow: $(q_k)_{t_1 t_2} = \dots$.

- L_2 and the higher Lagrangians are closely related to the variational symmetries of L_1.

  [Petrera, Suris. Variational symmetries and pluri-Lagrangian systems in classical mechanics. In preparation.]

Example: Potential KdV hierarchy

$v_{t_2} = g_2[v] = v_{xxx} + 3v_x^2,$

$v_{t_3} = g_3[v] = v_{xxxxx} + 10v_x v_{xxx} + 5v_{xx}^2 + 10v_x^3,$

where we identify $t_1 = x$.

The differentiated equations $v_{xt_i} = \frac{\mathrm{d}}{\mathrm{d}x} g_i[v]$ are Lagrangian with

$$L_{12} = \frac{1}{2} v_x v_{t_2} - \frac{1}{2} v_x v_{xxx} - v_x^3,$$
$$L_{13} = \frac{1}{2} v_x v_{t_3} - v_x v_{xxxxx} - 2v_{xx} v_{xxxx} - \frac{3}{2} v_{xxx}^2 + 5v_x^2 v_{xxx} + 5v_x v_{xx}^2 + \frac{5}{2} v_x^4.$$

We choose the coefficient L_{23} of

$$\mathcal{L} = L_{12}[u]\,\mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}[u]\,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}[u]\,\mathrm{d}t_2 \wedge \mathrm{d}t_3$$

such that the pluri-Lagrangian 2-form is closed on solutions. It is of the form

$$L_{23} = \frac{1}{2}(v_{t_2} g_3[v] - v_{t_3} g_2[v]) + p_{23}[v].$$

Example: Potential KdV hierarchy

- The equations

$$\frac{\delta_{12} L_{12}}{\delta v} = 0 \qquad \text{and} \qquad \frac{\delta_{13} L_{13}}{\delta v} = 0$$

are

$$v_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x} g_2[v] \qquad \text{and} \qquad v_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x} g_3[v].$$

- The equations

$$\frac{\delta_{12} L_{12}}{\delta v_x} = \frac{\delta_{32} L_{32}}{\delta v_{t_3}} \qquad \text{and} \qquad \frac{\delta_{13} L_{13}}{\delta v_x} = \frac{\delta_{23} L_{23}}{\delta v_{t_2}}$$

yield

$$v_{t_2} = g_2 \qquad \text{and} \qquad v_{t_3} = g_3,$$

the evolutionary equations!

- All other multi-time EL equations are corollaries of these.

Relation to Hamiltonian formalism

Consider a pluri-Lagrangian two form $\sum_{i,j} L_{ij} \mathrm{d}t_i \wedge \mathrm{d}t_j$ with

$$L_{1j} = \frac{1}{2} v_x v_{t_j} - h_j(v_x, v_{xx}, \dots)$$

and L_{ij} such that the multi-time Euler-Lagrange equations are

$$v_{t_j} = g_j(v_x, v_{xx}, \dots) \qquad \text{with } g_j = \frac{\delta_1 h_j}{\delta v_x}$$

Introducing the variable $u = v_x$ we can write this as

$$u_{t_j} = \mathrm{D}_x g_j(u, u_x, \dots). \qquad \text{(cf. KdV)}$$

This equation is Hamiltonian with Hamilton function h_j w.r.t. the Poisson bracket

$$\left\{ \int f, \int g \right\} = \int \left(D_x \frac{\delta_1 f}{\delta u} \right) \frac{\delta_1 g}{\delta u}$$

on equivalence classes $(\int \cdot)$ mod x-derivatives.

If the pluri-Lagrangian two form $\sum_{i,j} L_{ij} \mathrm{d}t_i \wedge \mathrm{d}t_j$ is closed on solutions, then the Hamiltonians are in involution: $\left\{ \int h_i, \int h_j \right\} = 0$

Relation to variational symmetries

[Petrera, Suris. Variational symmetries and pluri-Lagrangian systems in classical mechanics. In preparation.]

Consider a mechanical Lagrangian $L(q, q_t)$.

We say that a (generalized) vector field $V(q, q_t)$ is a variational symmetry if there exists a function $F(q, q_t)$, called the flux, such that

$$\mathrm{D}_V L(q, q_t) - \mathrm{D}_t F(q, q_t) = 0.$$

Noether's Theorem

If $V(q, q_t)$ is a variational symmetry with flux $F(q, q_t)$, then

$$J(q, q_t) = \frac{\partial L(q, q_t)}{\partial q_t} \cdot V(q, q_t) - F(q, q_t)$$

is an integral of motion.

Relation to variational symmetries

If we have a variational symmetry V with flux F and Noether integral J, then there is a pluri-Lagrangian one-form

$$\mathcal{L} = L_1(q, q_{t_1}, q_{t_2})\,\mathrm{d}t_1 + L_2(q, q_{t_1}, q_{t_2})\,\mathrm{d}t_2$$

with

$$\begin{aligned} L_1(q, q_{t_1}, q_{t_2}) &= L(q, q_{t_1}) \\ L_2(q, q_{t_1}, q_{t_2}) &= \frac{\partial L(q, q_{t_1})}{\partial q_{t_1}} \cdot q_{t_2} - J(q, q_{t_1}) \\ &= \frac{\partial L(q, q_{t_1})}{\partial q_{t_1}} \cdot (q_{t_2} - V(q, q_{t_1})) + F(q, q_{t_1}) \end{aligned}$$

which produces the equations of motion

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_1}\frac{\partial L}{\partial q_{t_1}} = 0 \qquad \text{and} \qquad q_{t_2} = V(q, q_{t_1})$$

If we have k commuting variational symmetries, we can produce a pluri-Lagrangian system in $k+1$ dimensions.

Open question: continuum limits

Form the discrete KdV equation

$$(x - x_{12})(x_2 - x_1) = \alpha_1 - \alpha_2 \qquad \text{(dKdV)}$$

the whole KdV hierarchy can be obtained using clever continuum limits.

Can we relate the discrete pluri-Lagrangian structure of (dKdV) with the continuous pluri-Lagrangian structure of the KdV hierarchy?

More general, is there a 1 to 1 correspondence between discrete and continuous pluri-Lagrangian systems?

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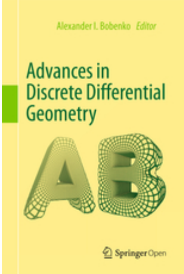
Main:

- Suris, Vermeeren. On the Lagrangian structure of integrable hierarchies.
  In Advances in Discrete Differential Geometry, Springer 2016.



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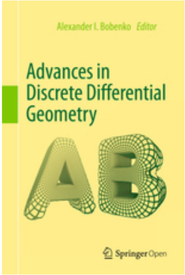
Discretize the theory, not just the equations

References



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