Continuum limits of integrable lattice equations

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Liouville-Arnold integrability

A Hamiltonian system with Hamilton function $H : \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville-Arnold integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \dots H_N$ such that $\{H_i, H_j\} = 0$.

- each H_i is a conserved quantity for all flows.
- the flows commute:

  The equations $\frac{\mathrm{d}}{\mathrm{d}t_i} z = \{H_i, z\}$ have simultaneous solutions $z(t_1, \dots, t_N)$.

  Solutions defined on multi-time $\mathbb{R}^N$ with coordinates $(t_1, \dots, t_N)$.

Integrable PDEs

Infinite degrees of freedom

$\to$ infinite hierarchy of commuting flows required for integrability.

Often constructed by a recursion operator, which in turn is often obtained form a bi-Hamiltonian structure.

Discrete counterpart of commuting flows

- Commuting maps $F_i : Q \to Q$,

$$F_i \circ F_j = F_j \circ F_i$$

  Often, the F_i only differ by a parameter:

$$F_i(z) = G(z, \lambda_i)$$

- Multidimensional consistency:

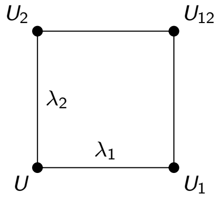
  There exists simultaneous solutions $z : \mathbb{N}^N \to Q$ to

$$z(k_1, \ldots, k_j + 1, \ldots, k_N) = F_j(z(k_1, \ldots, k_j, \ldots, k_N))$$

  The parameters λ_i can be associated to the directions of the lattice $\mathbb{N}^N$

Integrable quad equations

$$\mathcal{Q}(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2) = 0$$

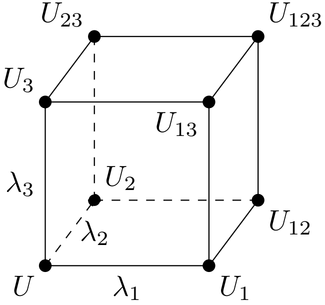


Subscripts of U denote lattice shifts, λ_1, λ_2 are parameters.
Invariant under symmetries of the square, affine in each of U, U_1, U_2, U_{12}.

Integrability for systems quad equations:
Multi-dimensional consistency of

$$\mathcal{Q}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j) = 0,$$

i.e. the three ways of calculating U_{123} give the same result.



Classification (under some extra assumptions) by Adler, Bobenko and Suris (ABS List).

Example: lattice potential KdV:
$(U - U_{12})(U_1 - U_2) - \lambda_1 + \lambda_2 = 0$

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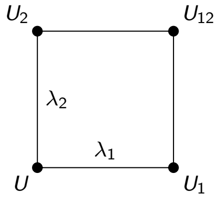

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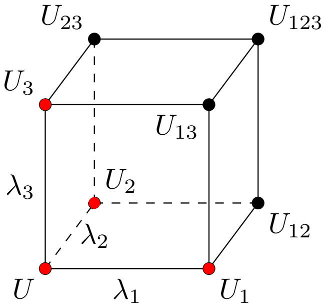

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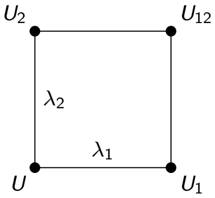

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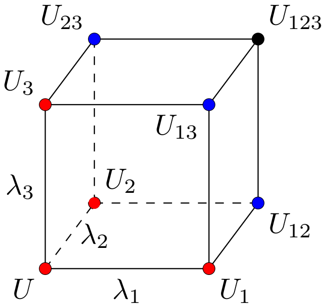

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$(U - U_{12})(U_1 - U_2) - \lambda_1 + \lambda_2 = 0$

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Continuum limits

Naive continuum limits often destroy the dynamics

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Magic ingredient 1: Miwa shifts

Skew embedding of the mesh $\mathbb{Z}^N$ into multi-time $\mathbb{R}^N$

Discrete U is a sampling of the continuous u:

$$U = U(\mathbf{n}) = u(t_1, t_2, \ldots, t_N),$$

$$U_i = U(\mathbf{n} + \mathfrak{e}_i) = u\left(t_1 - 2\lambda_i, t_2 + 2\frac{\lambda_i^2}{2}, \ldots, t_N + 2(-1)^N \frac{\lambda_i^N}{N} \right)$$

[Miwa. On Hirota's difference equations. Proceedings of the Japan Academy A. 1982]

Example: lpKdV

$$(U - U_{12})(U_2 - U_1) = \lambda_1^2 - \lambda_2^2$$

$$\Rightarrow \; 4(\lambda_1 + \lambda_2)u_{t_1}(\lambda_1 - \lambda_2)u_{t_1} + \mathcal{O}(\lambda_i^3) = \lambda_1^2 - \lambda_2^2$$

$$\Rightarrow \; u_{t_1}^2 = \tfrac{1}{4}$$

$\rightarrow$ ODE in t_1 instead of PDE.

Continuum limit of H1 (lattice potential KdV)

Magic ingredient 2: a smart transformation

In the case of lpKdV, the transformation
$U(m,n) \mapsto U(m,n) + m\lambda_1 + n\lambda_2$, followed by $\lambda_i \mapsto \lambda_i^{-1}$ yields

$$\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + U_{1,2} - U \right) \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} + U_2 - U_1 \right) = \frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2}$$

We get cancellation of leading order terms in λ_1, λ_2 of this equation when using the Miwa shifts

$$U_i = U(\mathbf{n} + \mathfrak{e}_i) = u\left(t_1 - 2\lambda_i, t_2 + 2\frac{\lambda_i^2}{2}, \ldots, t_n + 2(-1)^n \frac{\lambda_i^N}{N} \right).$$

$\Rightarrow$ The first nonvanishing term involves derivatives w.r.t. t_2 as well as t_1.

$\Rightarrow$ We get PDEs

Commutativity of the PDEs follows from multidimensional consistency of the quad equation.

Continuum limit of H1 (lattice potential KdV)

Series expansion

$$\text{Quad Equation} \quad \rightarrow \quad \sum_{i,j} \frac{4}{ij} f_{i,j}[u]\, \lambda_1^i \lambda_2^j = 0,$$

where $f_{j,i} = -f_{i,j}$ and the factor $\dfrac{4}{ij}$ is chosen to normalize the $f_{0,j}$.

First row of coefficients:

$$\begin{aligned}
f_{0,1} &= -u_{t_2}, \\
f_{0,2} &= -3u_{t_1}^2 - u_{t_1 t_1 t_1} - \tfrac{3}{2} u_{t_1 t_2} + u_{t_3}, \\
f_{0,3} &= 8u_{t_1} u_{t_1 t_1} + 4u_{t_1} u_{t_2} + \tfrac{4}{3} u_{t_1 t_1 t_1 t_1} - \tfrac{4}{3} u_{t_1 t_3} - u_{t_2 t_2} - u_{t_4}, \\
f_{0,4} &= -5u_{t_1 t_1}^2 - \tfrac{20}{3} u_{t_1} u_{t_1 t_1 t_1} + 10 u_{t_1} u_{t_1 t_2} + 5 u_{t_1 t_1} u_{t_2} - \tfrac{5}{4} u_{t_2}^2 - \tfrac{10}{3} u_{t_1} u_{t_3} \\
&\qquad - u_{t_1 t_1 t_1 t_1 t_1} + \tfrac{5}{3} u_{t_1 t_1 t_1 t_2} + \tfrac{5}{4} u_{t_1 t_2 t_2} - \tfrac{5}{4} u_{t_1 t_4} - \tfrac{5}{3} u_{t_2 t_3} + u_{t_5}, \\
&\vdots
\end{aligned}$$

Continuum limit of H1 (lattice potential KdV)

Setting each f_{ij} equal to zero, we find

$$
\begin{aligned}
u_{t_2} &= 0, \\
u_{t_3} &= 3u_{t_1}^2 + u_{t_1 t_1 t_1} \\
u_{t_4} &= 0, \\
u_{t_5} &= 10u_{t_1}^3 + 5u_{t_1 t_1}^2 + 10 u_{t_1} u_{t_1 t_1 t_1} + u_{t_1 t_1 t_1 t_1 t_1}, \\
&\dots
\end{aligned}
$$

$\hookrightarrow$ pKdV hierarchy

Whole hierarchy from single quad equation

using Miwa correspondence

$$U_i = U(\mathbf{n} + \mathfrak{e}_i) = u\left(t_1 - 2\lambda_i, t_2 + 2\frac{\lambda_i^2}{2}, \dots, t_n + 2(-1)^n \frac{\lambda_i^N}{N}\right)$$

Multidimensional consistency $\leftrightarrow$ commuting PDEs

[Equivalent continuum limit for lpKdV already by Wiersma and Capel in 1987]

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Variational analogue of $\{H_i, H_j\} = 0$

Integrable systems come in families: finite (for ODEs) or infinite (for PDEs) hierarchies of commuting equations.

In addition, on the Hamiltonian side, integrability is characterized by $\{H_i, H_j\} = 0$.

What about the Lagrangian side?

Variational analogue of $\{H_i, H_j\} = 0$

Integrable systems come in families: finite (for ODEs) or infinite (for PDEs) hierarchies of commuting equations.

In addition, on the Hamiltonian side, integrability is characterized by $\{H_i, H_j\} = 0$.

What about the Lagrangian side?

Pluri-Lagrangian principle / Lagrangian multiform principle

Combine the Lagrange functions $L_i[u]$ into a Lagrangian 1-form

$$\mathcal{L}[u] = \sum_i L_i[u] \, \mathrm{d}t_i.$$

Look for dynamical variables $u(t_1, \ldots, t_N)$ such that the action

$$S_\Gamma = \int_\Gamma \mathcal{L}[u]$$

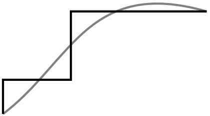
is critical w.r.t. variations of u, simultaneously over every curve Γ in multi-time $\mathbb{R}^N$

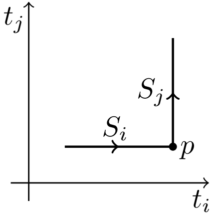
Multi-time Euler-Lagrange equations

Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i[u] \, \mathrm{d}t_i$

Lemma

If the action $\int_S \mathcal{L}$ is critical on all stepped curves S in $\mathbb{R}^N$, then it is critical on all smooth curves.

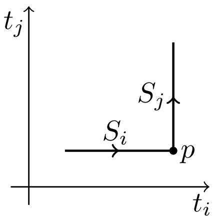




Variations are local, so it is sufficient to look at a general L-shaped curve $S = S_i \cup S_j$.

Multi-time Euler-Lagrange equations

$$\delta \int_{S_i} L_i \,\mathrm{d}t_i = \int_{S_i} \sum_I \frac{\partial L_i}{\partial u_I} \delta u_I \,\mathrm{d}t_i$$
$$= \int_{S_i} \sum_{I \not\ni t_i} \sum_{\alpha=0}^{\infty} \frac{\partial L_i}{\partial u_{I t_i^\alpha}} \delta u_{I t_i^\alpha} \,\mathrm{d}t_i$$
$$= \int_{S_i} \sum_{I \not\ni t_i} \frac{\delta_i L_i}{\delta u_I} \delta u_I \,\mathrm{d}t_i + \sum_I \frac{\delta_i L_i}{\delta u_{I t_i}} \delta u_I \bigg|_p ,$$



where I denotes a multi-index, and

$$\frac{\delta_i L_i}{\delta u_I} = \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial L_i}{\partial u_{I t_i^\alpha}} = \frac{\partial L_i}{\partial u_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial u_{I t_i}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial L_i}{\partial u_{I t_i^2}} - \dots$$

Multi-time Euler-Lagrange equations for curves, $\mathcal{L} = \sum_i L_i[u] \,\mathrm{d}t_i$

$$\frac{\delta_i L_i}{\delta u_I} = 0 \qquad \forall I \not\ni t_i \qquad \text{and} \qquad \frac{\delta_i L_i}{\delta u_{I t_i}} = \frac{\delta_j L_j}{\delta u_{I t_j}} \qquad \forall I,$$

Example: Kepler Problem

The classical Lagrangian of a particle in the gravitational potential

$$L_1[q] = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

can be combined with

$$L_2[q] = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot \mathbf{e},$$

($\mathbf{e}$ a fixed vector) into a pluri-Lagrangian 1-form

$$L_1 \, \mathrm{d}t_1 + L_2 \, \mathrm{d}t_2.$$

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$$L_1 \,\mathrm{d}t_1 + L_2 \,\mathrm{d}t_2.$$

Multi-time Euler-Lagrange equations:

$$\frac{\delta_1 L_1}{\delta q} = 0 \quad \Rightarrow \quad q_{t_1 t_1} = -\frac{q}{|q|^3} \qquad \text{(Keplerian motion)}$$

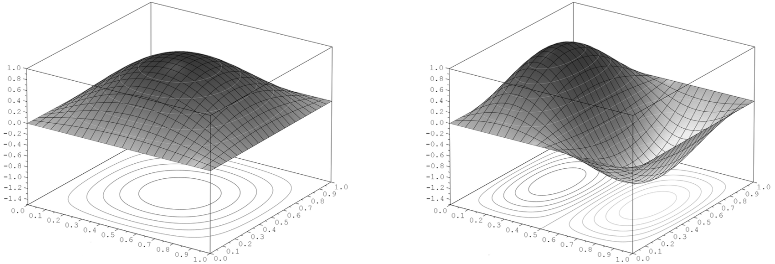
$$\frac{\delta_2 L_2}{\delta q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_2} = e \times q \qquad \text{(Rotation)}$$

Pluri-Lagrangian principle for 2-dimensional PDEs

Given a 2-form

$$\mathcal{L} = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

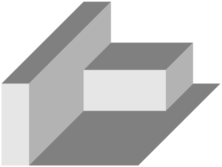
find a field $u : \mathbb{R}^N \to \mathbb{C}$, such that $\int_\Gamma \mathcal{L}$ is critical on all smooth 2-surfaces Γ in multi-time $\mathbb{R}^N$, w.r.t. variations of u.



Example: KdV hierarchy, where $t_1 = x$ is the shared space coordinate, t_i time for i-th flow. (Details to follow.)

Multi-time EL equations

Consider a Lagrangian 2-form $\mathcal{L} = \sum_{i,j} L_{ij}[u]\,\mathrm{d}t_i \wedge \mathrm{d}t_j$.

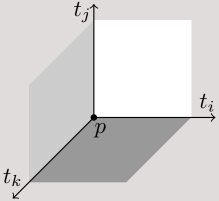


Every smooth surface can be approximated arbitrarily well by stepped surfaces. Hence it is sufficient to require criticality on stepped surfaces, or just on their elementary corners.

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{It_k t_i}} = 0 \qquad \forall I.$$



Where

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial u_{I t_i^\alpha t_j^\beta}}$$

Example: Potential KdV hierarchy

Identify $t_1 = x$ and consider the 2-form $\sum_{i<j} L_{ij} \mathrm{d}t_i \wedge \mathrm{d}t_j$ with

$$
\begin{aligned}
L_{12} &= \tfrac{1}{2} u_x u_{t_2} - \tfrac{1}{2} u_x u_{xxx} - u_x^3, \\
L_{13} &= \tfrac{1}{2} u_x u_{t_3} - \tfrac{1}{2} u_{xxx}^2 + 5 u_x u_{xx}^2 - \tfrac{5}{2} u_x^4. \\
L_{23} &= u_x^5 - \tfrac{15}{2} u_x^2 u_{xx}^2 + 10 u_x^3 u_{xxx} - 5 u_x^3 u_{t_2} + \tfrac{7}{2} u_{xx}^2 u_{xxx} + 3 u_x u_{xxx}^2 \\
&\quad - 6 u_x u_{xx} u_{xxxx} + \tfrac{3}{2} u_x^2 u_{xxxxx} + 10 u_x u_{xx} u_{xt_2} - \tfrac{5}{2} u_{xx}^2 u_{t_2} - 5 u_x u_{xxx} u_{t_2} \\
&\quad + \tfrac{3}{2} u_x^2 u_{t_3} - \tfrac{1}{2} u_{xxxx}^2 + \tfrac{1}{2} u_{xxx} u_{xxxxx} - u_{xxx} u_{xxt_2} + \tfrac{1}{2} u_x u_{xxt_3} \\
&\quad + u_{xxxx} u_{xt_2} - \tfrac{1}{2} u_{xx} u_{xt_3} - \tfrac{1}{2} u_{xxxxx} u_{t_2} + \tfrac{1}{2} u_{xxx} u_{t_3}
\end{aligned}
$$

Example: Potential KdV hierarchy

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&\quad + \tfrac{3}{2} u_x^2 u_{t_3} - \tfrac{1}{2} u_{xxxx}^2 + \tfrac{1}{2} u_{xxx} u_{xxxxx} - u_{xxx} u_{xxt_2} + \tfrac{1}{2} u_x u_{xxt_3} \\
&\quad + u_{xxxx} u_{xt_2} - \tfrac{1}{2} u_{xx} u_{xt_3} - \tfrac{1}{2} u_{xxxxx} u_{t_2} + \tfrac{1}{2} u_{xxx} u_{t_3}
\end{aligned}
$$

The classical EL equations yield

$$
u_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(3u_x^2 + u_{xxx}\right) \quad \text{and} \quad u_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x}\left(10u_x^3 + 5u_{xx}^2 + 10 u_x u_{xxx} + u_{xxxxx}\right).
$$

The multi-time EL equations yield

$$
u_{t_2} = 3u_x^2 + u_{xxx} \qquad \text{and} \qquad u_{t_3} = 10u_x^3 + 5u_{xx}^2 + 10 u_x u_{xxx} + u_{xxxxx}.
$$

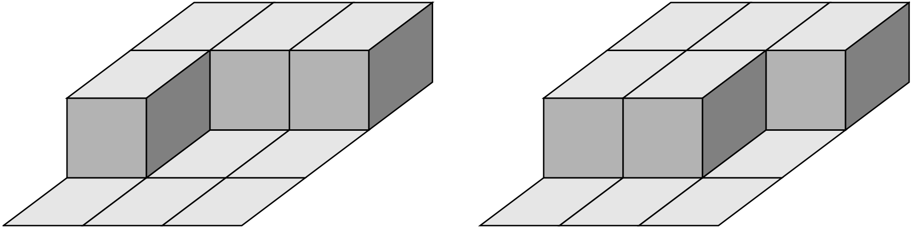
The pluri-Lagrangian structure produces evolutionary equations!

Pluri-Lagrangian structure for quad equations

For some discrete 2-form

$$\mathcal{L}(\square_{ij}) = \mathcal{L}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j),$$

the action $\displaystyle\sum_{\square \in \Gamma} \mathcal{L}(\square)$ is critical on all 2-surfaces Γ in $\mathbb{N}^N$ simultaneously.



To derive Euler-Lagrange equations: vary U at each point individually.

$\hookrightarrow$ It is sufficient to consider corners of an elementary cube.

[Lobb, Nijhoff. 2009]

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Continuum limit of a Lagrangian 2-form

$$L(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2)$$

Suitable choice $\Rightarrow$ leading order cancellation

$\downarrow$ Miwa shifts, Taylor expansion

$$\mathcal{L}_{\mathrm{disc}}([u], \lambda_1, \lambda_2)$$

Action is still a sum

$\downarrow$ Euler-Maclaurin formula

$$\mathcal{L}_{\mathrm{Miwa}}([u], \lambda_1, \lambda_2)$$

With $L(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2) = \int \mathcal{L}_{\mathrm{Miwa}}([u(\mathbf{t})], \lambda_1, \lambda_2)\, \eta_1 \wedge \eta_2$,
where η_1 and η_2 are the 1-forms dual to the Miwa shifts.

$\downarrow$

$$\sum_{1 \leq i < j \leq N} \mathcal{L}_{\mathrm{Miwa}}([u], \lambda_i, \lambda_j)\, \eta_i \wedge \eta_j$$

Continuum limit of a Lagrangian 2-form

$L(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2)$ Suitable choice $\Rightarrow$ leading order cancellation

$$\Big\downarrow \text{ Miwa shifts, Taylor expansion}$$

$\mathcal{L}_{\mathrm{disc}}([u], \lambda_1, \lambda_2)$ Action is still a sum

$$\Big\downarrow \text{ Euler-Maclaurin formula}$$

$$\mathcal{L}_{\mathrm{Miwa}}([u], \lambda_1, \lambda_2) = \sum_{i,j=1}^{\infty} (-1)^{i+j} 4 \frac{\lambda_1^i}{i} \frac{\lambda_2^j}{j} \mathcal{L}_{i,j}[u]$$

With $L(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2) = \int \mathcal{L}_{\mathrm{Miwa}}([u(\mathbf{t})], \lambda_1, \lambda_2)\, \eta_1 \wedge \eta_2$,
where η_1 and η_2 are the 1-forms dual to the Miwa shifts.

$$\Big\downarrow \qquad\qquad\qquad\qquad \Big\downarrow$$

$$\sum_{1 \le i < j \le N} \mathcal{L}_{\mathrm{Miwa}}([u], \lambda_i, \lambda_j)\, \eta_i \wedge \eta_j = \sum_{1 \le i < j \le N} \mathcal{L}_{i,j}[u]\, \mathrm{d}t_i \wedge \mathrm{d}t_j$$

Continuum limit of the Lagrangian for H1

Lagrangian for (lpKdV)

$$L(\square) = \frac{1}{2}\left(U - U_{i,j} - \lambda_i^{-1} - \lambda_j^{-1}\right)\left(U_i - U_j + \lambda_i^{-1} - \lambda_j^{-1}\right) + \left(\lambda_i^{-2} - \lambda_j^{-2}\right)\log\left(1 + \frac{U_i - U_j}{\lambda_i^{-1} - \lambda_j^{-1}}\right).$$

A well-chosen representative among many equivalent Lagrangians.

Continuum limit procedure:

- Miwa correspondence:

$$U = U(\mathbf{n}) = u(t_1, t_2, \ldots, t_n),$$
$$U_i = U(\mathbf{n} + \mathfrak{e}_i) = u\left(t_1 - 2\lambda_i, t_2 + 2\frac{\lambda_i^2}{2}, \ldots, t_n + 2(-1)^n \frac{\lambda_i^N}{N}\right)$$

- Series expansion
- Euler-Maclaurin formula

Coefficients (after some post-limit simplifications)

$$\mathcal{L}_{1,2} = \tfrac{1}{2} u_1 u_2 \qquad \mathcal{L}_{1,3} = -u_1^3 + \tfrac{1}{2} u_{11}^2 + \tfrac{1}{2} u_1 u_3$$

$$\mathcal{L}_{1,4} = \tfrac{1}{2} u_1 u_4 \qquad \mathcal{L}_{1,5} = -\tfrac{5}{2} u_1^4 + 5 u_1 u_{11}^2 - \tfrac{1}{2} u_{111}^2 + \tfrac{1}{2} u_1 u_5$$

$$\mathcal{L}_{2,3} = -3 u_1^2 u_2 + u_{11} u_{12} - u_{111} u_2 + \tfrac{1}{2} u_2 u_3$$

$$\mathcal{L}_{2,4} = \tfrac{1}{2} u_2 u_4$$

$$\mathcal{L}_{2,5} = -10 u_1^3 u_2 + 10 u_1 u_{11} u_{12} - 5 u_{11}^2 u_2 - 10 u_1 u_{111} u_2 - u_{111} u_{112} + u_{1111} u_{12} - u_{11111} u_2 + \tfrac{1}{2} u_2 u_5$$

$$\mathcal{L}_{3,4} = -u_{11} u_{14} + \tfrac{1}{2} u_3 u_4$$

$$\mathcal{L}_{3,5} = 18 u_1^5 + 30 u_1^3 u_{111} - 10 u_1^3 u_3 + 6 u_{11}^2 u_{111} + 8 u_1 u_{111}^2 - 6 u_1 u_{11} u_{1111} + 3 u_1^2 u_{11111} + 10 u_1 u_{11} u_{13} - 5 u_{11}^2 u_3 - 10 u_1 u_{111} u_3 - \tfrac{1}{2} u_{1111}^2 + u_{111} u_{11111} - u_{111} u_{113} + u_{1111} u_{13} - u_{11} u_{15} - u_{11111} u_3 + \tfrac{1}{2} u_3 u_5$$

$$\mathcal{L}_{4,5} = -10 u_1^3 u_4 + 10 u_1 u_{11} u_{14} - 5 u_{11}^2 u_4 - 10 u_1 u_{111} u_4 - u_{111} u_{114} + u_{1111} u_{14} - u_{11111} u_4 + \tfrac{1}{2} u_4 u_5$$

Coefficients (after some post-limit simplifications)

$\mathcal{L}_{1,2} = \frac{1}{2} u_1 u_2$ $\qquad$ $\mathcal{L}_{1,3} = -u_1^3 + \frac{1}{2} u_{11}^2 + \frac{1}{2} u_1 u_3$

$\mathcal{L}_{1,4} = \frac{1}{2} u_1 u_4$ $\qquad$ $\mathcal{L}_{1,5} = -\frac{5}{2} u_1^4 + 5 u_1 u_{11}^2 - \frac{1}{2} u_{111}^2 + \frac{1}{2} u_1 u_5$

$$\mathcal{L}_{2,3} = -3u_1^2 u_2 + u_{11} u_{12} - u_{111} u_2 + \frac{1}{2} u_2 u_3$$

$$\mathcal{L}_{2,4} = \frac{1}{2} u_2 u_4$$

$$\begin{aligned} \mathcal{L}_{2,5} = {} & -10 u_1^3 u_2 + 10 u_1 u_{11} u_{12} - 5 u_{11}^2 u_2 - 10 u_1 u_{111} u_2 - u_{111} u_{112} + \\ & u_{1111} u_{12} - u_{11111} u_2 + \frac{1}{2} u_2 u_5 \end{aligned}$$

$$\mathcal{L}_{3,4} = -u_{11} u_{14} + \frac{1}{2} u_3 u_4$$

$$\begin{aligned} \mathcal{L}_{3,5} = {} & 18 u_1^5 + 30 u_1^3 u_{111} - 10 u_1^3 u_3 + 6 u_{11}^2 u_{111} + 8 u_1 u_{111}^2 - 6 u_1 u_{11} u_{1111} + \\ & 3 u_1^2 u_{11111} + 10 u_1 u_{11} u_{13} - 5 u_{11}^2 u_3 - 10 u_1 u_{111} u_3 - \frac{1}{2} u_{1111}^2 + \\ & u_{111} u_{11111} - u_{111} u_{113} + u_{1111} u_{13} - u_{11} u_{15} - u_{11111} u_3 + \frac{1}{2} u_3 u_5 \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{4,5} = {} & -10 u_1^3 u_4 + 10 u_1 u_{11} u_{14} - 5 u_{11}^2 u_4 - 10 u_1 u_{111} u_4 - u_{111} u_{114} + \\ & u_{1111} u_{14} - u_{11111} u_4 + \frac{1}{2} u_4 u_5 \end{aligned}$$

Continuum limits of ABS equations

$\mathrm{Q1}_{\delta=0}$	$\to$	$v_3 = v_{111} - \dfrac{3v_{11}^2}{2v_1}$	Schwarzian KdV
$\mathrm{Q1}_{\delta=1}$	$\to$	$v_3 = v_{111} - \dfrac{3}{2}\dfrac{v_{11}^2 - \frac{1}{4}}{v_1}$	
Q2	$\to$	$v_3 = v_{111} - \dfrac{3}{2}\dfrac{v_{11}^2 - \frac{1}{4}}{v_1} - \dfrac{3}{2}\dfrac{v_1^3}{v^2}$	
$\mathrm{Q3}_{\delta=0}$	$\to$	$v_3 = v_{111} - \dfrac{3}{2}\dfrac{v_{11}^2 - \frac{1}{4}}{v_1} + \dfrac{1}{2}v_1^3$	
$\mathrm{Q3}_{\delta=1}$	$\to$	$v_3 = v_{111} - \dfrac{3}{2}\dfrac{v_{11}^2 - \frac{1}{4}}{v_1} + \dfrac{1}{2}v_1^3 - \dfrac{3}{2}\dfrac{v_1^3}{\sin(v)^2}$	
Q4	$\to$	$v_3 = v_{111} - \dfrac{3}{2}\dfrac{v_{11} - \frac{1}{4}}{v_1} - \dfrac{3}{2}\wp(2v)v_1^3$	Krichever-Novikov
H1	$\to$	$v_3 = v_{111} + 3v_1^2$	Potential KdV
$\mathrm{H3}_{\delta=0}$	$\to$	$v_3 = v_{111} + \dfrac{1}{2}v_1^3$	Potential mKdV

All with their hierarchies and with a pluri-Lagrangian structure.

Conclusions

- Integrable continuum limits can be performed in a partially algorithmic way
  - Leading order cancellation needs to be obtained by hand
  - All the rest is guaranteed by Miwa shifts
- One multi-dimensionally consistent quad equation leads to an infinite hierarchy of PDEs.
  - One integrable PDE carries less information than one quad equation.
  - Is this why integrable discretization is difficult?
- No Hamiltonian structure for lattice equations.
  - Variational structures apply to both discrete and continuous equations.
  - Pluri-Lagrangian structures as a framework for studying discretization and continuum limit.

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Continuum limits of pluri-Lagrangian systems

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Thank you for your attention!