Pluri-Lagrangian systems

(a.k.a. Lagrangian multiform systems)

Mats Vermeeren

Technische Universität Berlin

June 8, 2017

ISQS 25, Prague

Discretization in Geometry and Dynamics SFB Transregio 109

Berlin Mathematical School

Main question

Many integrable systems (Toda lattice, KdV equation,...) come with infinite hierarchies. Each individual equation is Lagrangian/Hamiltonian.

On the Hamiltonian side, a compatibility condition is easy to state: $\{H_i, H_j\} = 0$.

What about the Lagrangian side?

Is there a variational description of an integrable hierarchy?

1. Discrete pluri-Lagrangian systems

2. Continuous pluri-Lagrangian systems

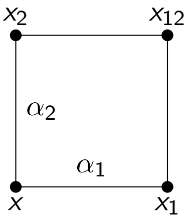
3. Relations
   - to Hamiltonian formalism
   - between continuous and discrete
   - to variational symmetries

Quad equations

Quad equation on $\mathbb{Z}^2$:

$$Q(x, x_1, x_2, x_{12}, \alpha_1, \alpha_2) = 0$$

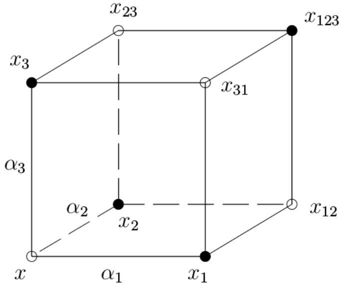
Subscripts of x denote lattice shifts, α_1, α_2 are parameters.
Invariant under symmetries of the square, affine in each of x, x_1, x_2, x_{12}.



Integrability for systems quad equations:
Multi-dimensional consistency of

$$Q(x, x_i, x_j, x_{ij}, \alpha_i, \alpha_j) = 0,$$

i.e. the three ways of calculating x_{123} give the same result.



Quad equations

- Classification multidimensionally consistent quad equations in the ABS list.

  [VE Adler, AI Bobenko, YB Suris. Classification of integrable equations on quad-graphs. The consistency approach. Commun. Math. Phys. 2003.]

- Variational formulation in which the Lagrangian is "an extended object capable of producing a multitude of consistent equations"
  $\hookrightarrow$ i.e. defined in the higher-dimensional lattice

  [S Lobb, F Nijhoff. Lagrangian multiforms and multidimensional consistency. J. Phys. A. 2009.]

Pluri-Lagrangian problem

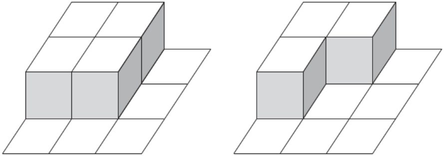
For some discrete 2-form

$$\mathcal{L}(\sigma_{ij}) = \mathcal{L}(x, x_i, x_j, x_{ij}, \alpha_i, \alpha_j),$$

find a field $x : \mathbb{Z}^N \to \mathbb{C}$ such that the action

$$\sum_{\sigma_{ij} \in S} \mathcal{L}(\sigma_{ij})$$

is critical on all discrete 2-surfaces S in $\mathbb{Z}^N$ simultaneously.



Furthermore, the critical value of the action does not depend on the surface, i.e. the discrete 2-form $\mathcal{L}$ is closed on solutions.

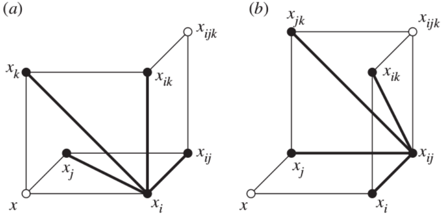
Example: H1 (lattice potential KdV)

$$(x - x_{ij})(x_i - x_j) - \alpha_i + \alpha_j = 0$$

we have the Lagrangian

$$\mathcal{L}(x, x_i, x_j, x_{ij}, \alpha_i, \alpha_j) = (x_i - x_j)x - (\alpha_i - \alpha_i) \log(x_i - x_j)$$

All surfaces can be build out of elementary corners:



$$(a) \quad x_{ij} - x_{ik} - \frac{\alpha_i - \alpha_k}{x_i - x_k} + \frac{\alpha_i - \alpha_j}{x_i - x_j} = 0,$$

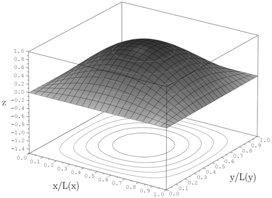
$$(b) \quad x_i - x_j - \frac{\alpha_j - \alpha_k}{x_{ij} - x_{ik}} + \frac{\alpha_i - \alpha_k}{x_{ij} - x_{jk}} = 0$$

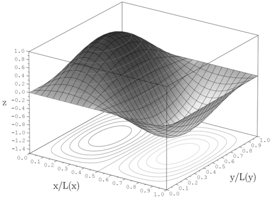
Pluri-Lagrangian 2-forms

Given a 2-form

$$\mathcal{L} = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

find a field $u : \mathbb{R}^N \to \mathbb{C}$, such that $\displaystyle\int_\Gamma \mathcal{L}$ is critical on all smooth 2-surfaces Γ in multi-time $\mathbb{R}^N$.





Possible for any dimension: given a d-form

$$\mathcal{L} = \sum_{i_1,\dots,i_d} L_{i_1 \dots i_d}[u] \, \mathrm{d}t_{i_1} \wedge \dots \wedge \mathrm{d}t_{i_d}.$$

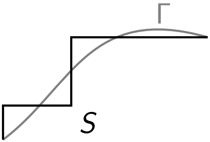
find u such that $\int_\Gamma \mathcal{L}$ is critical on all smooth d-surfaces Γ in $\mathbb{R}^N$.

Warm-up: $d = 1$

Consider a Lagrangian 1-form $\mathcal{L} = \sum_i L_i[u]\,\mathrm{d}t_i$

Lemma

If the action $\int_S \mathcal{L}$ is critical on all stepped curves S in $\mathbb{R}^N$, then it is critical on all smooth curves.

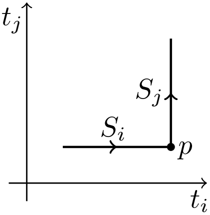


Proof

$$\int_\Gamma \mathcal{L} - \int_S \mathcal{L} = \int_M \mathrm{d}\mathcal{L},$$

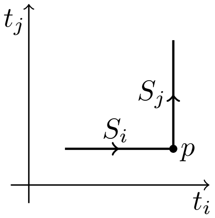
where $\partial M = \Gamma \cup S$. By choosing a fine approximation, M can be made arbitrarily small. □

Variations are local, so it is sufficient to look at a general L-shaped curve $S = S_i \cup S_j$.



Multi-time EL equations for the 1-form case

The variation of the action on S_i is



$$\delta \int_{S_i} L_i \,\mathrm{d}t_i = \int_{S_i} \sum_I \frac{\partial L_i}{\partial u_I} \delta u_I \,\mathrm{d}t_i$$
$$= \int_{S_i} \sum_{I \not\ni t_i} \frac{\delta_i L_i}{\delta u_I} \delta u_I \,\mathrm{d}t_i + \sum_I \frac{\delta_i L_i}{\delta u_{I t_i}} \delta u_I \bigg|_p ,$$

where I denotes a multi-index, and

$$\frac{\delta_i L_i}{\delta u_I} = \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial L_i}{\partial u_{I t_i^\alpha}} = \frac{\partial L_i}{\partial u_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial u_{I t_i}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial L_i}{\partial u_{I t_i^2}} - \dots$$

Multi-time Euler-Lagrange equations for $\mathcal{L} = \sum_i L_i[u] \,\mathrm{d}t_i$

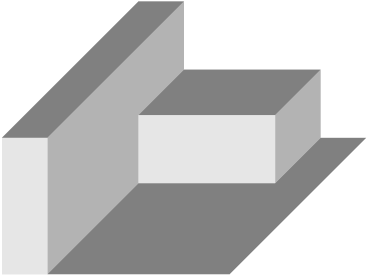
$$\frac{\delta_i L_i}{\delta u_I} = 0 \qquad \forall I \not\ni t_i \qquad \text{and} \qquad \frac{\delta_i L_i}{\delta u_{I t_i}} = \frac{\delta_j L_j}{\delta u_{I t_j}} \qquad \forall I,$$

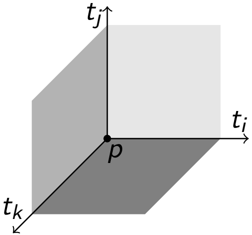
$d = 2$

Consider a Lagrangian 2-form

$$\mathcal{L} = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j .$$

It is sufficient to look at stepped surfaces and their elementary corners.

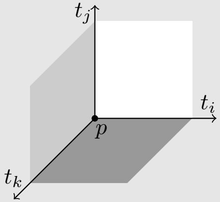




$d = 2$

Multi-time EL equations for $\mathcal{L} = \sum_{i,j} L_{ij}[u]\,\mathrm{d}t_i \wedge \mathrm{d}t_j$

$$
\begin{aligned}
&\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 && \forall I \not\ni t_i, t_j,\\
&\frac{\delta_{ij} L_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{It_k}} && \forall I \not\ni t_i,\\
&\frac{\delta_{ij} L_{ij}}{\delta u_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{It_k t_i}} = 0 && \forall I.
\end{aligned}
$$



Where

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial u_{It_i^\alpha t_j^\beta}}$$

[YB Suris, MV. On the Lagrangian structure of integrable hierarchies. In AI Bobenko (ed): Advances in Discrete Differential Geometry, Springer. 2016.]

Example: Potential KdV hierarchy

$$u_{t_2} = g_2[u] = u_{xxx} + 3u_x^2,$$
$$u_{t_3} = g_3[u] = u_{xxxxx} + 10u_x u_{xxx} + 5u_{xx}^2 + 10u_x^3,$$

where we identify $t_1 = x$.

The differentiated equations $u_{xt_i} = \frac{\mathrm{d}}{\mathrm{d}x} g_i[u]$ are Lagrangian with

$$L_{12} = \frac{1}{2}u_x u_{t_2} - \frac{1}{2}u_x u_{xxx} - u_x^3,$$
$$L_{13} = \frac{1}{2}u_x u_{t_3} - u_x u_{xxxxx} - 2u_{xx}u_{xxxx} - \frac{3}{2}u_{xxx}^2 + 5u_x^2 u_{xxx} + 5u_x u_{xx}^2 + \frac{5}{2}u_x^4.$$

But we also need a coefficient L_{23} to define a 2-form

$$\mathcal{L} = L_{12}[u]\,\mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}[u]\,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}[u]\,\mathrm{d}t_2 \wedge \mathrm{d}t_3$$

Example: Potential KdV hierarchy

We choose L_{23} such that

$$\mathcal{L} = L_{12}[u]\,\mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}[u]\,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}[u]\,\mathrm{d}t_2 \wedge \mathrm{d}t_3$$

is closed on solutions. We can take it in the form

$$L_{23} = \frac{1}{2}(u_{t_2} g_3 - u_{t_3} g_2) + (a_{23} - a_{32}) - \frac{1}{2}(b_{23} - b_{32}).$$

where

- $a_{ij} := u_{t_j} \frac{\delta_1 h_i}{\delta u_x} + u_{xt_j} \frac{\delta_1 h_i}{\delta u_{xx}} + u_{xxt_j} \frac{\delta_1 h_i}{\delta u_{xxx}} + \ldots$, where $h_i = \frac{1}{4i+2} g_{i+1}$.
- b_{ij} is a polynomial in $u, u_x, u_{xx}, \ldots$ such that $\mathrm{D}_x b_{ij} = g_j \mathrm{D}_x g_i$.

The closedness condition implies that

- the action $\int_S \mathcal{L}$ is independent of perturbations of the surface S,
- the flows are variational symmetries of each other.

Example: Potential KdV hierarchy

- The equations

$$\frac{\delta_{12} L_{12}}{\delta u} = 0 \qquad \text{and} \qquad \frac{\delta_{13} L_{13}}{\delta u} = 0$$

are

$$u_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x} g_2[u] \qquad \text{and} \qquad u_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x} g_3[u].$$

- The equations

$$\frac{\delta_{12} L_{12}}{\delta u_x} = \frac{\delta_{32} L_{32}}{\delta u_{t_3}} \qquad \text{and} \qquad \frac{\delta_{13} L_{13}}{\delta u_x} = \frac{\delta_{23} L_{23}}{\delta u_{t_2}}$$

yield

$$u_{t_2} = g_2 \qquad \text{and} \qquad u_{t_3} = g_3,$$

- All other multi-time EL equations are corollaries of these.

1. Discrete pluri-Lagrangian systems

2. Continuous pluri-Lagrangian systems

3. Relations
   - to Hamiltonian formalism
   - between continuous and discrete
   - to variational symmetries

Relation to Hamiltonian formalism

Consider a pluri-Lagrangian two form $\sum_{i,j} L_{ij} \mathrm{d}t_i \wedge \mathrm{d}t_j$ with

$$L_{1j} = \frac{1}{2} u_x u_{t_j} - h_{1j}(u_x, u_{xx}, \ldots)$$

and such that the multi-time Euler-Lagrange equations are

$$u_{t_j} = g_j(u_x, u_{xx}, \ldots) \qquad \text{with } g_j = \frac{\delta_1 h_{1j}}{\delta u_x}$$

This equation is Hamiltonian with Hamilton function h_{1j} w.r.t. the Gardner-Zakharov-Faddeev Poisson bracket

$$\{F, G\}_1 = -\frac{\delta_1 F}{\delta u} \frac{\delta_1 G}{\delta u_x} = \frac{\delta_1 F}{\delta u} \mathrm{D}_x^{-1} \frac{\delta_1 G}{\delta u}$$

on equivalence classes modulo x-derivatives.

Relation to Hamiltonian formalism

In fact there is a whole family of brackets:

$$\{F, G\}_i = \frac{\delta_i F}{\delta u} \mathrm{D}_i^{-1} \frac{\delta_i G}{\delta u}$$

on equivalence classes modulo $t_1, \ldots, t_N$-derivatives.

For

$$h_{ij} = \frac{1}{2} u_{t_i} u_{t_j} - L_{ij},$$

there holds

$$\forall i, j : \quad u_{t_j} = \{u, h_{ij}\}_i$$

If the pluri-Lagrangian two form $\sum_{i,j} L_{ij} \mathrm{d}t_i \wedge \mathrm{d}t_j$ is closed on solutions, then the Hamiltonians are "in involution":

$$\{h_{ij}, h_{ik}\}_i = 0 \quad \forall i, j, k$$

[Work in progress]

Discrete meets continuous

Quad equations like the lattice pKdV have two interpretations.

$$(U - U_{ij})(U_i - U_j) - \lambda_i + \lambda_j = 0, \qquad U = U(n_1, \ldots, n_N).$$

	Discrete	**Continuous**
U	dependent variable	dependent variable
$n_i \in \mathbb{Z}$	independent variables	parameters
$\lambda_i \in \mathbb{R}$	paramters	independent variables

But there is also a different way to pass from discrete to continuous,

		Continuum limit	
U		u	dependent variable
n_i	$\to$	t_j	independent variables
λ_i		–	no parameters

which yields a hierarchy of differential equations for $u(t_1, \ldots, t_N)$.

Continuum limits

From H1 (lattice pKdV) the whole pKdV hierarchy can be obtained.

[GL Wiersma, HW Capel. Lattice equations, hierarchies and Hamiltonian structures. Physica A. 1987]

The pluri-Lagrangian structure survives this limit:

discrete Lagrangian 2-form $\rightarrow$ continuous Lagrangian 2-form

The same is true for Q1 (discrete cross-ratio equation), which produces the Schwarzian KdV hierarchy.

[MV. Continuum limits of pluri-Lagrangian systems. in preparation]

Is there a 1 to 1 correspondence between discrete and continuous pluri-Lagrangian systems?

Relation to variational symmetries

Consider a mechanical Lagrangian $L(q, q_t)$.

We say that a (generalized) vector field $V(q, q_t)$ is a variational symmetry if there exists a function $F(q, q_t)$, called the flux, such that

$$\mathrm{D}_V L(q, q_t) - \mathrm{D}_t F(q, q_t) = 0.$$

Noether's Theorem

If $V(q, q_t)$ is a variational symmetry with flux $F(q, q_t)$, then

$$J(q, q_t) = \frac{\partial L(q, q_t)}{\partial q_t} V(q, q_t) - F(q, q_t)$$

is an integral of motion.

Relation to variational symmetries

If we have a variational symmetry V with flux F and Noether integral J, then there is a pluri-Lagrangian 1-form

$$\mathcal{L} = L_1(q, q_{t_1}, q_{t_2})\,\mathrm{d}t_1 + L_2(q, q_{t_1}, q_{t_2})\,\mathrm{d}t_2$$

with

$$\begin{aligned} L_1(q, q_{t_1}, q_{t_2}) &= L(q, q_{t_1}) \\ L_2(q, q_{t_1}, q_{t_2}) &= \frac{\partial L(q, q_{t_1})}{\partial q_{t_1}}(q_{t_2} - V(q, q_{t_1})) + F(q, q_{t_1}) \end{aligned}$$

which produces the equations of motion

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_1}\frac{\partial L}{\partial q_{t_1}} = 0 \qquad \text{and} \qquad q_{t_2} = V(q, q_{t_1}).$$

Can be extended to more dimensions if more variational symmetries exist.

[M Petrera, YB Suris. Variational symmetries and pluri-Lagrangian systems in classical mechanics. In preparation.]

Quantization

- Discrete action for $d = 1 \to$ propagator in multi-time

  Worked out for harmonic oscillator in:

  [SD King, FW Nijhoff. Quantum Variational Principle and quantum multiform structure: the case of quadratic Lagrangians. arXiv:1702.08709.]

- Continuous 1-form case $\to$ path integrals in multi-time ?

- QFT ?

Selected references

- VE Adler, AI Bobenko, YB Suris. Classification of integrable equations on quad-graphs. The consistency approach. Commun. Math. Phys. 2003.
- S Lobb, F Nijhoff. Lagrangian multiforms and multidimensional consistency. J. Phys. A. 2009.
- R Boll, M Petrera, YB Suris. What is integrability of discrete variational systems? Proc. R. Soc. A. 2014.
- J Hietarinta, N Joshi, FW Nijhoff. Discrete Systems and Integrability. (Chapter 12) Cambridge Texts in Applied Mathematics. 2016.
- YB Suris, MV. On the Lagrangian structure of integrable hierarchies. In AI Bobenko (ed): Advances in Discrete Differential Geometry, Springer. 2016.
- SD King, FW Nijhoff. Quantum Variational Principle and quantum multiform structure: the case of quadratic Lagrangians. `arXiv:1702.08709`.
- MV. Continuum limits of pluri-Lagrangian systems. in preparation