Continuum limits of pluri-Lagrangian systems

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July 6, 2017

Discretization in Geometry and Dynamics SFB Transregio 109

Berlin Mathematical School

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Motivation 1: variational analogue of $\{H_i, H_j\} = 0$

Integrable systems come in families: either finite (classical mechanics, ...) or infinite (Toda lattice, KdV equation,...) hierarchies of commuting equations. Each individual equation is Lagrangian/Hamiltonian.

On the Hamiltonian side, integrability is characterized by $\{H_i, H_j\} = 0$.

What about the Lagrangian side?

Motivation 1: variational analogue of $\{H_i, H_j\} = 0$

Integrable systems come in families: either finite (classical mechanics, ...) or infinite (Toda lattice, KdV equation,...) hierarchies of commuting equations. Each individual equation is Lagrangian/Hamiltonian.

On the Hamiltonian side, integrability is characterized by $\{H_i, H_j\} = 0$.

What about the Lagrangian side?

Pluri-Lagrangian principle ($d = 1$)

Combine the Lagrange functions $\mathcal{L}_i$ into a Lagrangian 1-form

$$\mathcal{L} = \sum_i \mathcal{L}_i \, \mathrm{d}t_i.$$

Look for fields $u : \mathbb{R}^N \to \mathbb{C}$ that minimize the action

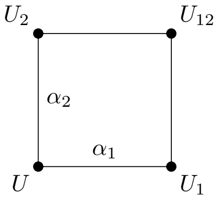
$$S_\Gamma = \int_\Gamma \mathcal{L}$$

simultaneously over every curve Γ in multi-time $\mathbb{R}^N$

Motivation 2: understanding quad equations

Quad equation on $\mathbb{Z}^2$:

$$Q(U, U_1, U_2, U_{12}, \alpha_1, \alpha_2) = 0$$



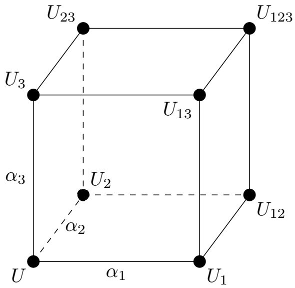
Subscripts of U denote lattice shifts, α_1, α_2 are parameters.

Q invariant under symmetries of the square, affine in each of U, U_1, U_2, U_{12}.

Integrability for systems quad equations: Multi-dimensional consistency of

$$Q(U, U_i, U_j, U_{ij}, \alpha_i, \alpha_j) = 0,$$

i.e. the three ways of calculating U_{123} give the same result.



Motivation 2: understanding quad equations

- Classification multidimensionally consistent quad equations in the ABS list.

  [Adler, Bobenko, Suris. Classification of integrable equations on quad-graphs. The consistency approach. Commun. Math. Phys. 2003.]

- Variational formulation in which the Lagrangian is “an extended object capable of producing a multitude of consistent equations”
  $\hookrightarrow$ i.e. defined in the higher-dimensional lattice

  [Lobb, Nijhoff. Lagrangian multiforms and multidimensional consistency. J. Phys. A. 2009.]

Pluri-Lagrangian principle ($d = 2$, discrete)

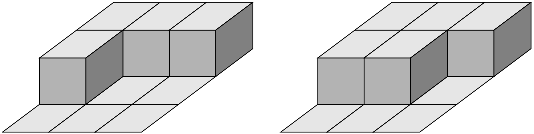
For some discrete 2-form

$$\mathcal{L}(\sigma_{ij}) = \mathcal{L}(U, U_i, U_j, U_{ij}, \alpha_i, \alpha_j),$$

find a field $U : \mathbb{Z}^N \to \mathbb{C}$ such that the action

$$\textstyle\sum_{\sigma_{ij} \in S} \mathcal{L}(\sigma_{ij})$$

is critical on all discrete 2-surfaces S in $\mathbb{Z}^N$ simultaneously.



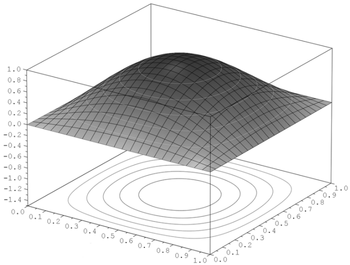
- EL equations obtained from corners of cubes.
- All ABS equations can be described this way.

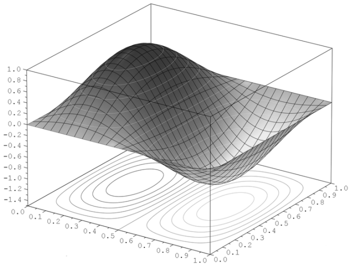
Pluri-Lagrangian principle ($d = 2$, continuous)

Given a 2-form

$$\mathcal{L} = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

find a field $u : \mathbb{R}^N \to \mathbb{C}$, such that $\int_\Gamma \mathcal{L}$ is critical on all smooth 2-surfaces Γ in multi-time $\mathbb{R}^N$.



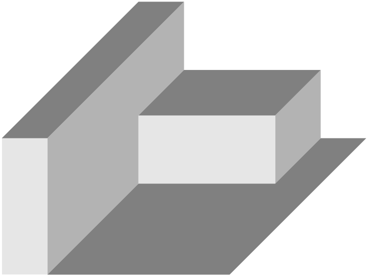


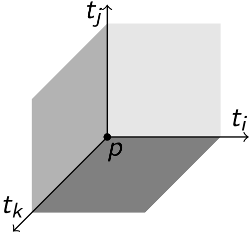
How to calculate Euler-Lagrange equations? Unlike the discrete case there are no elementary building blocks of smooth surfaces.

Multi-time EL equations

Consider a Lagrangian 2-form $\mathcal{L} = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$.

Every smooth surface can be approximated arbitrarily well by stepped surfaces. Hence it is sufficient to require criticality on stepped surfaces, or just on their elementary corners.





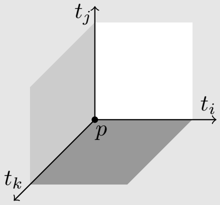
Multi-time EL equations

for $\mathcal{L} = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{It_k t_i}} = 0 \qquad \forall I.$$



Where

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial u_{It_i^\alpha t_j^\beta}}$$

[Suris, V. On the Lagrangian structure of integrable hierarchies. In AI Bobenko (ed): Advances in Discrete Differential Geometry, Springer. 2016.]

Continuum limit of H1 (lattice potential KdV)

$$\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + U_{1,2} - U\right)\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} + U_2 - U_1\right) = \frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2} \quad \text{(lpKdV)}$$

This is a well-chosen representative of H1 out of many equivalent forms.

Often one finds it written as $(X - X_{12})(X_2 - X_1) = \alpha_1 - \alpha_2$

Method by Wiersma and Capel produces the pKdV hierarchy from (lpKdV)

They used an differential-difference equation as intermediate step. Here we will present the same limit in a single step.

[Wiersma, Capel. Lattice equations, hierarchies and Hamiltonian structures. Physica A. 1987]

Continuum limit of H1 (lattice potential KdV)

Miwa shifts

Skew embedding of the mesh $\mathbb{Z}^N$ into multi-time $\mathbb{R}^N$
Discrete $U : \mathbb{Z}^N \to \mathbb{C}$ is a sampling of the continuous $u : \mathbb{R}^N \to \mathbb{C}$:

$$U = U(\mathbf{n}) = u(t_1, t_2, \ldots, t_n),$$
$$U_i = U(\mathbf{n} + \mathfrak{e}_i) = u\left(t_1 - 2\lambda_i, t_2 + 2\frac{\lambda_i^2}{2}, \ldots, t_n + 2(-1)^n \frac{\lambda_i^N}{N}\right)$$

[Miwa. On Hirota's difference equations. Proceedings of the Japan Academy A. 1982]

Plug into $\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + U_{1,2} - U\right)\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} + U_2 - U_1\right) = \frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2}$
and expand in λ_1, λ_2.

In leading order everything cancels (due to very specific form of quad eq).
$\hookrightarrow$ generically we would have an ODE in t_1.

Continuum limit of H1 (lattice potential KdV)

Series expansion

$$\text{Quad Equation} \quad \rightarrow \quad \sum_{i,j} \frac{4}{ij} f_{i,j}[u] \, \lambda_1^i \lambda_2^j = 0,$$

where $f_{j,i} = -f_{i,j}$ and the factor $\frac{4}{ij}$ is chosen to normalize the $f_{0,j}$.

First row of coefficients:

$$f_{0,1} = -u_{t_2},$$

$$f_{0,2} = -3u_{t_1}^2 - u_{t_1 t_1 t_1} - \frac{3}{2} u_{t_1 t_2} + u_{t_3},$$

$$f_{0,3} = 8u_{t_1} u_{t_1 t_1} + 4u_{t_1} u_{t_2} + \frac{4}{3} u_{t_1 t_1 t_1 t_1} - \frac{4}{3} u_{t_1 t_3} - u_{t_2 t_2} - u_{t_4},$$

$$f_{0,4} = -5u_{t_1 t_1}^2 - \frac{20}{3} u_{t_1} u_{t_1 t_1 t_1} + 10u_{t_1} u_{t_1 t_2} + 5u_{t_1 t_1} u_{t_2} - \frac{5}{4} u_{t_2}^2 - \frac{10}{3} u_{t_1} u_{t_3}$$

$$\vdots$$

Continuum limit of H1 (lattice potential KdV)

Setting each f_{ij} equal to zero, we find

$$
\begin{aligned}
u_{t_2} &= 0, \\
u_{t_3} &= 3u_{t_1}^2 + u_{t_1 t_1 t_1} \\
u_{t_4} &= 0, \\
u_{t_5} &= 10u_{t_1}^3 + 5u_{t_1 t_1}^2 + 10u_{t_1} u_{t_1 t_1 t_1} + u_{t_1 t_1 t_1 t_1 t_1}, \\
&\vdots
\end{aligned}
$$

$\hookrightarrow$ pKdV hierarchy

Whole hierarchy from single quad equation

using Miwa correspondence

$$
\begin{aligned}
U &= U(\mathbf{n}) = u(t_1, t_2, \ldots, t_n), \\
U_i &= U(\mathbf{n} + \mathfrak{e}_i) = u\left(t_1 - 2\lambda_i, t_2 + 2\frac{\lambda_i^2}{2}, \ldots, t_n + 2(-1)^n \frac{\lambda_i^N}{N} \right)
\end{aligned}
$$

Continuum limit of the Lagrangian for H1

A Lagrangian for (lpKdV) is

$$L(\square) = \frac{1}{2}\left(U - U_{i,j} - \lambda_i^{-1} - \lambda_j^{-1}\right)\left(U_i - U_j + \lambda_i^{-1} - \lambda_j^{-1}\right) + \left(\lambda_i^{-2} - \lambda_j^{-2}\right)\log\left(1 + \frac{U_i - U_j}{\lambda_i^{-1} - \lambda_j^{-1}}\right).$$

Again this is a specific (and non-standard) choice among the many equivalent Lagrangians.

Using Miwa correspondence:

$$\text{Discrete } L \quad \to \quad \text{Power series } \mathcal{L}_{\mathrm{disc}}$$

Continuum limit of the Lagrangian

A series expansion is not the end of the story here. The action would still be a sum:

$$S = \sum_{\square \in \sigma} L(\square) \;=\; \sum_{\square \in \sigma} \mathcal{L}_{\mathrm{disc}}[u(\text{point in } \square)].$$

We want an integral

$$S = \int_\sigma \mathcal{L}.$$

Euler-MacLaurin formula

$$\sum_{j=0}^{n-1} g(j) = \int_0^n g(t)\,\mathrm{d}t + \sum_{i=1}^{\infty} \frac{B_i}{i!}\Big(g^{(i-1)}(n) - g^{(i-1)}(0)\Big)$$

$$= \int_0^n \left(g(t) + \sum_{i=1}^{\infty} \frac{B_i}{i!} g^{(i)}(t)\right)\mathrm{d}t,$$

where the B_i are the Bernoulli numbers $1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \cdots$.

Continuum limit of the Lagrangian

$$\mathcal{L}_{\mathrm{Miwa}}([u],\lambda_1,\lambda_2) = \sum_{i,j=0}^{\infty} \frac{B_i B_j}{i!j!} \partial_1^i \partial_2^j \mathcal{L}_{\mathrm{disc}}([u],\lambda_1,\lambda_2).$$

where the differential operators are $\quad \partial_k = \sum_{j=1}^{N} (-1)^{j+1} \frac{\lambda_k^j}{j} \frac{\mathrm{d}}{\mathrm{d}t_j}$.

Then there holds

$$L_{\mathrm{disc}}(\square_{1,2}) = \int_{\blacksquare_{1,2}} \mathcal{L}_{\mathrm{Miwa}}([u(\mathbf{t})],\lambda_1,\lambda_2).$$

Theorem

Write $\quad \mathcal{L}_{\mathrm{Miwa}}([u],\lambda_1,\lambda_2) = \sum_{i,j=1}^{\infty} \frac{\lambda_1^i}{i} \frac{\lambda_2^j}{j} \mathcal{L}_{ij}[u],$

then $\quad \mathcal{L} = \sum_{1 \leq i < j \leq n} \mathcal{L}_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j \quad$ is a pluri-Lagrangian structure.

Continuum limit of the Lagrangian

Theorem

Write $\quad \mathcal{L}_{\mathrm{Miwa}}([u], \lambda_1, \lambda_2) = \sum_{i,j=1}^{\infty} \frac{\lambda_1^i}{i} \frac{\lambda_2^j}{j} \mathcal{L}_{ij}[u],$

then $\quad \mathcal{L} = \sum_{1 \leq i < j \leq n} \mathcal{L}_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j \quad$ is a pluri-Lagrangian structure.

Proof (sketch).

- In the 2-form, $\mathcal{L}_{ij}[u]$ corresponds to t_i and t_j,
- t_i and t_j correspond to λ_1^i and λ_2^j under Miwa shifts.

Discrete and continuous action agree:

$$\int_\Gamma \mathcal{L} = \sum_{\Gamma_{\mathrm{disc}}} L(\square) \quad \text{if } \Gamma_{\mathrm{disc}} \mapsto \Gamma \text{ under the Miwa correspondence.}$$

These Γ form a sufficiently large class of surfaces to derive the multi-time EL equations.

Coefficients (after some post-limit simplifications)

$$\mathcal{L}_{1,2} = \tfrac{1}{2} u_1 u_2 \qquad \mathcal{L}_{1,3} = -u_1^3 + \tfrac{1}{2} u_{11}^2 + \tfrac{1}{2} u_1 u_3$$

$$\mathcal{L}_{1,4} = \tfrac{1}{2} u_1 u_4 \qquad \mathcal{L}_{1,5} = -\tfrac{5}{2} u_1^4 + 5 u_1 u_{11}^2 - \tfrac{1}{2} u_{111}^2 + \tfrac{1}{2} u_1 u_5$$

$$\mathcal{L}_{2,3} = -3u_1^2 u_2 + u_{11} u_{12} - u_{111} u_2 + \tfrac{1}{2} u_2 u_3$$

$$\mathcal{L}_{2,4} = \tfrac{1}{2} u_2 u_4$$

$$\mathcal{L}_{2,5} = -10 u_1^3 u_2 + 10 u_1 u_{11} u_{12} - 5 u_{11}^2 u_2 - 10 u_1 u_{111} u_2 - u_{111} u_{112} + u_{1111} u_{12} - u_{11111} u_2 + \tfrac{1}{2} u_2 u_5$$

$$\mathcal{L}_{3,4} = -u_{11} u_{14} + \tfrac{1}{2} u_3 u_4$$

$$\mathcal{L}_{3,5} = 18 u_1^5 + 30 u_1^3 u_{111} - 10 u_1^3 u_3 + 6 u_{11}^2 u_{111} + 8 u_1 u_{111}^2 - 6 u_1 u_{11} u_{1111} + 3 u_1^2 u_{11111} + 10 u_1 u_{11} u_{13} - 5 u_{11}^2 u_3 - 10 u_1 u_{111} u_3 - \tfrac{1}{2} u_{1111}^2 + u_{111} u_{11111} - u_{111} u_{113} + u_{1111} u_{13} - u_{11} u_{15} - u_{11111} u_3 + \tfrac{1}{2} u_3 u_5$$

$$\mathcal{L}_{4,5} = -10 u_1^3 u_4 + 10 u_1 u_{11} u_{14} - 5 u_{11}^2 u_4 - 10 u_1 u_{111} u_4 - u_{111} u_{114} + u_{1111} u_{14} - u_{11111} u_4 + \tfrac{1}{2} u_4 u_5$$

Coefficients (after some post-limit simplifications)

$$\mathcal{L}_{1,2} = \tfrac{1}{2} u_1 u_2 \qquad \mathcal{L}_{1,3} = -u_1^3 + \tfrac{1}{2} u_{11}^2 + \tfrac{1}{2} u_1 u_3$$

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Other equations

Q1: $$\frac{U_2 - U}{\lambda_2}\frac{U_{1,2} - U_1}{\lambda_2} - \frac{U_1 - U}{\lambda_1}\frac{U_{1,2} - U_2}{\lambda_1} = 0$$

produces the Schwarzian KdV hierarchy

$$\frac{u_{t_2}}{u_{t_1}} = 0,$$

$$\frac{u_{t_3}}{u_{t_1}} = -\frac{3u_{t_1 t_1}^2}{2u_{t_1}^2} + \frac{u_{t_1 t_1 t_1}}{u_{t_1}},$$

$$\frac{u_{t_4}}{u_{t_1}} = 0,$$

$$\frac{u_5}{u_{t_1}} = -\frac{45u_{t_1 t_1}^4}{8u_{t_1}^4} + \frac{25u_{t_1 t_1}^2 u_{t_1 t_1 t_1}}{2u_{t_1}^3} - \frac{5u_{t_1 t_1 t_1}^2}{2u_{t_1}^2} - \frac{5u_{t_1 t_1} u_{t_1 t_1 t_1 t_1}}{u_{t_1}^2} + \frac{u_{t_1 t_1 t_1 t_1 t_1}}{u_{t_1}},$$

$$\vdots$$

including a pluri-Lagrangian structure

Other equations

- $\mathrm{H3}_{\delta=0}$: produces the potential modified KdV hierarchy

  But so far, no Lagrangian has been found that allows a series expansion.

- No results for other ABS equations at this moment.

- 1-form example:

  Fully discrete Toda lattice:

  $$\frac{1}{\lambda}\left(e^{\widetilde{Q}_k - Q_k} - e^{Q_k - \underset{\sim}{Q}_k}\right) + \lambda\left(e^{Q_k - \widetilde{Q}_{k-1}} - e^{\underset{\sim}{Q}_{k+1} - Q_k}\right) = 0,$$

  where $\widetilde{\cdot}$ and $\underset{\sim}{\cdot}$ denote forward and backward shifts.

  $\rightarrow$ Toda hierarchy

  $$\begin{aligned} (q_k)_{t_1 t_1} &= e^{q_{k+1} - q_k} - e^{q_k - q_{k-1}} \\ (q_k)_{t_2} &= ((q_k)_{t_1})^2 + e^{q_{k+1} - q_k} + e^{q_k - q_{k-1}} \\ &\vdots \end{aligned}$$

References

Main:

- V. Continuum limits of pluri-Lagrangian systems. arXiv:1706.06830

Further reading:

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