Hamiltonian and Lagrangian perspectives on integrable hierarchies

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Hamiltonian Systems

Hamilton function $H : T^*Q \to \mathbb{R} : (q,p) \mapsto H(q,p)$ determines dynamics:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Poisson bracket of two functionals on T^*Q:

$$\{f,g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$

Dynamics of a Hamiltonian system:

$$\dot{q}_i = \{q_i, H\}, \qquad \dot{p}_i = \{p_i, H\}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} f(q,p) = \{f(q,p), H\}$$

Properties:

anti-symmetry: $\{f,g\} = -\{g,f\}$

bilinearity: $\{f, g + \lambda h\} = \{f,g\} + \lambda\{f,h\}$

Leibniz property: $\{f, gh\} = \{f,g\}h + g\{f,h\}$

Jacobi identity: $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

Liouville-Arnold integrability

A Hamiltonian system with Hamilton function $H : \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville-Arnold integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \ldots H_N$ such that $\{H_i, H_j\} = 0$.

- Each H_i is a conserved quantity for all flows.
- The dynamics is confined to a leaf of the foliation $\{H_i = \mathrm{const}\}$.
- The flows commute.
- There exists a symplectic change of variables $(p, q) \mapsto (\bar{p}, \bar{q})$ such that

$$H(p, q) = \bar{H}_i(\bar{p})$$

  Liouville-Arnold integrable systems evolve linearly in these variables!

  $(\bar{p}, \bar{q})$ are called action-angle variables.

Variational analogue of $\{H_i, H_j\} = 0$

Integrable systems come in families: finite (for ODEs) or infinite (for PDEs) hierarchies of commuting equations:

$$\frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\mathrm{d}}{\mathrm{d}t_j} = \frac{\mathrm{d}}{\mathrm{d}t_j}\frac{\mathrm{d}}{\mathrm{d}t_i} \qquad \text{for time variables } t_1, t_2, \ldots$$

On the Hamiltonian side, integrability is characterized by $\{H_i, H_j\} = 0$.

What about the Lagrangian side?

Variational analogue of $\{H_i, H_j\} = 0$

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What about the Lagrangian side?

Pluri-Lagrangian (Lagrangian multi-form) principle for ODEs

Combine the Lagrange functions $L_i[u]$ into a Lagrangian 1-form

$$\mathcal{L}[u] = \sum_i L_i[u]\,\mathrm{d}t_i.$$

Look for dynamical variables $u(t_1, \ldots, t_N)$ such that the action

$$S_\Gamma = \int_\Gamma \mathcal{L}[u]$$

is critical w.r.t. variations of u, simultaneously over every curve Γ in multi-time $\mathbb{R}^N$

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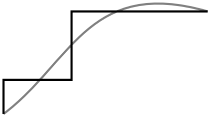
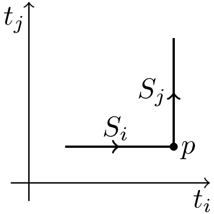
Multi-time Euler-Lagrange equations

Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i[u] \,\mathrm{d}t_i$

Lemma

If the action $\int_S \mathcal{L}$ is critical on all stepped curves S in $\mathbb{R}^N$, then it is critical on all smooth curves.

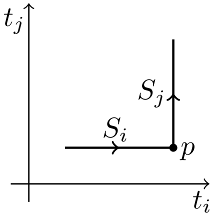




Variations are local, so it is sufficient to look at a general L-shaped curve $S = S_i \cup S_j$.

Multi-time Euler-Lagrange equations

$$\delta \int_{S_i} L_i \,\mathrm{d}t_i = \int_{S_i} \sum_I \frac{\partial L_i}{\partial u_I} \delta u_I \,\mathrm{d}t_i$$
$$= \int_{S_i} \sum_{I \not\ni t_i} \sum_{\alpha=0}^{\infty} \frac{\partial L_i}{\partial u_{It_i^\alpha}} \delta u_{It_i^\alpha} \,\mathrm{d}t_i$$
$$= \int_{S_i} \sum_{I \not\ni t_i} \frac{\delta_i L_i}{\delta u_I} \delta u_I \,\mathrm{d}t_i + \sum_I \frac{\delta_i L_i}{\delta u_{It_i}} \delta u_I \bigg|_p ,$$



where I denotes a multi-index, and

$$\frac{\delta_i L_i}{\delta u_I} = \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial L_i}{\partial u_{It_i^\alpha}} = \frac{\partial L_i}{\partial u_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial u_{It_i}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial L_i}{\partial u_{It_i^2}} - \dots$$

Multi-time Euler-Lagrange equations for curves, $\mathcal{L} = \sum_i L_i[u] \,\mathrm{d}t_i$

$$\frac{\delta_i L_i}{\delta u_I} = 0 \qquad \forall I \not\ni t_i \qquad \text{and} \qquad \frac{\delta_i L_i}{\delta u_{It_i}} = \frac{\delta_j L_j}{\delta u_{It_j}} \qquad \forall I,$$

Example: Kepler Problem

The classical Lagrangian of a particle in the gravitational potential

$$L_1[q] = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

can be combined with

$$L_2[q] = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot e,$$

into a pluri-Lagrangian 1-form $L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$ and consider $q = q(t_1, t_2)$.

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Multi-time Euler-Lagrange equations:

$$\frac{\delta_1 L_1}{\delta q} = 0 \quad \Rightarrow \quad q_{t_1 t_1} = -\frac{q}{|q|^3} \qquad \text{(Keplerian motion)}$$

$$\frac{\delta_2 L_2}{\delta q} = 0 \quad \Rightarrow \quad q_{t_1 t_2} = e \times q_{t_1}$$

$$\frac{\delta_2 L_2}{\delta q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_2} = e \times q \qquad \text{(Rotation)}$$

$$\frac{\delta_1 L_1}{\delta q_{t_1}} = \frac{\delta_2 L_2}{\delta q_{t_2}} \quad \Rightarrow \quad q_{t_1} = q_{t_1}$$

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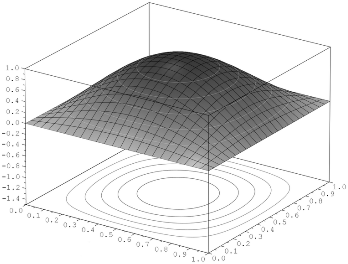
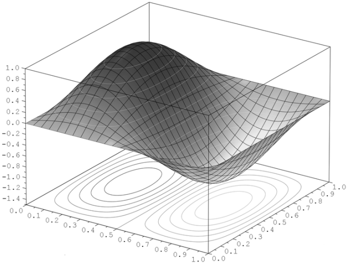
Pluri-Lagrangian principle ($d = 2$, continuous)

Given a 2-form

$$\mathcal{L} = \textstyle\sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

find a field $u : \mathbb{R}^N \to \mathbb{C}$, such that $\int_\Gamma \mathcal{L}$ is critical on all smooth 2-surfaces Γ in multi-time $\mathbb{R}^N$, w.r.t. variations of u.



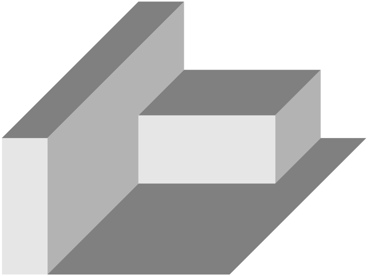


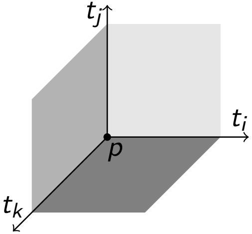
Example: KdV hierarchy, where $t_1 = x$ is the shared space coordinate, t_i time for i-th flow. (Details to follow.)

Multi-time EL equations

Consider a Lagrangian 2-form $\mathcal{L} = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$.

Every smooth surface can be approximated arbitrarily well by stepped surfaces. Hence it is sufficient to require criticality on stepped surfaces, or just on their elementary corners.





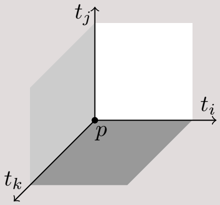
Multi-time EL equations

for $\mathcal{L} = \sum_{i,j} L_{ij}[u]\, \mathrm{d}t_i \wedge \mathrm{d}t_j$

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{It_k t_i}} = 0 \qquad \forall I.$$



Where

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial u_{I t_i^\alpha t_j^\beta}}$$

Example: Potential KdV hierarchy

$$u_{t_2} = Q_2 = u_{xxx} + 3u_x^2,$$
$$u_{t_3} = Q_3 = u_{xxxxx} + 10u_x u_{xxx} + 5u_{xx}^2 + 10u_x^3,$$
where we identify $t_1 = x$.

The differentiated equations $u_{xt_i} = \frac{\mathrm{d}}{\mathrm{d}x} Q_i$ are Lagrangian with

$$L_{12} = \frac{1}{2} u_x u_{t_2} - \frac{1}{2} u_x u_{xxx} - u_x^3,$$
$$L_{13} = \frac{1}{2} u_x u_{t_3} - u_x u_{xxxxx} - 2u_{xx} u_{xxxx} - \frac{3}{2} u_{xxx}^2 + 5u_x^2 u_{xxx} + 5u_x u_{xx}^2 + \frac{5}{2} u_x^4.$$

Example: Potential KdV hierarchy

$$u_{t_2} = Q_2 = u_{xxx} + 3u_x^2,$$
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A suitable coefficient L_{23} of

$$\mathcal{L} = L_{12}\,\mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}\,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}\,\mathrm{d}t_2 \wedge \mathrm{d}t_3$$

can be found (nontrivial task!) in the form

$$L_{23} = \frac{1}{2}(u_{t_2} Q_3 - u_{t_3} Q_2) + p_{23}.$$

Example: Potential KdV hierarchy

- The equations $\frac{\delta_{12} L_{12}}{\delta u} = 0$ and $\frac{\delta_{13} L_{13}}{\delta u} = 0$ yield

$$u_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x} Q_2 \qquad \text{and} \qquad u_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x} Q_3.$$

- The equations $\frac{\delta_{12} L_{12}}{\delta u_x} = \frac{\delta_{32} L_{32}}{\delta u_{t_3}}$ and $\frac{\delta_{13} L_{13}}{\delta u_x} = \frac{\delta_{23} L_{23}}{\delta u_{t_2}}$ yield

$$u_{t_2} = Q_2 \qquad \text{and} \qquad u_{t_3} = Q_3,$$

  the evolutionary equations!
- All other multi-time EL equations are corollaries of these.

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Hamiltonian structure of Lagrangian 1-form systems

Lagrangian 1-form systems and systems of commuting Hamiltonian flows are in 1-to-1 correspondence

[Suris, Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms. J. Geometric Mechanics, 2013]

Switching perspectives by Legendre transform is not possible, because

$$\left| \frac{\partial^2 \mathcal{L}}{\partial v^2} \right| = 0$$

so that

$$\mathbb{F} : TQ \to T^*Q : (q, v) \mapsto \left(q, \frac{\partial \mathcal{L}}{\partial v} \right)$$

Alternative strategy: Dirac reduction leads to a (constrained) Hamiltonian formulation of a degenerate Lagrangian system.

Dirac reduction

We focus on Lagrangians that are linear in the velocities:

$$\mathcal{L}(q, q_t) = p(q)^T q_t - V(q)$$

Notation: $p : Q \to \mathbb{R}^N$ is a function of the positions.
π is a bundle coordinate of T^*Q.

We would like to define the Hamiltonian by $H \circ \mathbb{F} = q_t \dfrac{\partial \mathcal{L}}{\partial q_t} - \mathcal{L}$,

but this only specifies H on the image of the Legendre transform $\mathbb{F}(q, q_t) = (q, p(q))$:

$$\begin{aligned} H(q, p(q)) &= p(q)^T v - \mathcal{L}(q, q_t) \\ &= V(q) \end{aligned}$$

Let $H : T^*Q \to \mathbb{R} : (q, \pi) \to H(q, \pi)$ be any extension of this function and impose $\pi - p(q) = 0$ as a constraint in the variational principle:

$$\delta \int H(q, \pi) - \pi^T q_t - \lambda^T (\pi - p(q)) \, \mathrm{d}t = 0$$

Dirac reduction

$$\delta \int H(q,\pi) - \pi^T q_t - \lambda^T (\pi - p(q)) \,\mathrm{d}t = 0$$

Variations with respect to q, π, and λ yield

$$\begin{aligned} \pi_t &= -\frac{\partial H}{\partial q} - \lambda^T \frac{\partial p(q)}{\partial q}, \\ q_t &= \frac{\partial H}{\partial \pi} - \lambda, \\ \pi &= p(q). \end{aligned}$$

In terms of the canonical Poisson bracket

$$\{f,g\} = \frac{\partial f}{\partial \pi}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial \pi}\frac{\partial f}{\partial q}$$

on T^*Q, the evolution of a function $f : T^*Q \to \mathbb{R}$ is given by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{H - \lambda^T c, f\} = \{H, f\} - \lambda^T \{c, f\},$$

where $c = \pi - p(q)$.

Dirac bracket

Let $\mathcal{M} = \{c, c^T\}$ be the matrix with

$$\mathcal{M}_{ij} = \{c_i, c_j\} = \frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i}.$$

The Dirac bracket on T^*Q is given by

$$\{f, g\}^D = \{f, g\} + \{c^T, f\}\mathcal{M}^{-1}\{c, g\}.$$

Properties:

- The Dirac bracket $\{\cdot, \cdot\}^D$ is a weak Poisson bracket, i.e.:
  - It is bilinear, skew-symmetric, and satisfies the Leibniz rule.
  - The Jacobi identity holds on the constraint manifold $\{c = 0\} \subset T^*Q$.
- For any function $f : T^*Q \to \mathbb{R}$ there holds

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{H, f\}^D.$$

- The constraints are Casimir functions: $\forall f : T^*Q \to \mathbb{R} : \{c, f\}^D = 0$.

From pluri-Lagrangian to Hamiltonian systems

Consider a pluri-Lagrangian 1-form $\mathcal{L} = \sum_i \mathcal{L}_i \, \mathrm{d}t_i$ consisting of

$$\mathcal{L}_1(q, q_1) = \frac{1}{2}|q_1|^2 - V_1(q)$$

and

$$\mathcal{L}_i(q, q_1, q_i) = q_1^T q_i - V_i(q, q_1) \qquad \text{for } i \geq 2,$$

- Momenta $p = q_1$ have to agree due to the multi-time Euler-Lagrange equation
- The first Hamiltonian is found by Legendre transform:

$$H_1(q, \pi) = \frac{1}{2}|\pi|^2 + V_1(q)$$

- For $i \geq 2$ we consider $r = q_1$ as a second independent variable.

  The Lagrangians $\mathcal{L}_i(q, r, q_i, r_i) = r q_i - V_i(q, r)$ are degenerate, so we use Dirac reduction.

From pluri-Lagrangian to Hamiltonian systems

The momenta corresponding to $\mathcal{L}_i(q, r, q_i, r_i) = rq_i - V_i(q, r)$ are

$$p_q = \frac{\partial \mathcal{L}_i}{\partial q_i} = r \qquad \text{and} \qquad p_r = \frac{\partial \mathcal{L}_i}{\partial r_i} = 0.$$

$\Rightarrow$ constraints $c_q = c_r = 0$ with $\quad c_q = \pi_q - r \quad$ and $\quad c_r = \pi_r.$

With respect to the Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial \pi_q}\frac{\partial g}{\partial q} + \frac{\partial f}{\partial \pi_r}\frac{\partial g}{\partial r} - \frac{\partial g}{\partial \pi_q}\frac{\partial f}{\partial q} - \frac{\partial g}{\partial \pi_r}\frac{\partial f}{\partial r}$$

we have

$$\mathcal{M} = \begin{pmatrix} \{c_q, c_q\} & \{c_q, c_r\} \\ \{c_r, c_q\} & \{c_r, c_r\} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathcal{M}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Dirac bracket

$$\begin{aligned} \{f, g\}^D &= \{f, g\} - \begin{pmatrix} \{f, c_q\} \\ \{f, c_r\} \end{pmatrix}^T \mathcal{M}^{-1} \begin{pmatrix} \{c_q, g\} \\ \{c_r, g\} \end{pmatrix} \\ &= \{f, g\} + \{f, c_q\}\{c_r, g\} - \{f, c_r\}\{c_q, g\}. \end{aligned}$$

From pluri-Lagrangian to Hamiltonian systems

Restricted to functions of q and r only (independent of momenta), the Dirac bracket reduces to

$$\begin{aligned} \{f,g\}^D &= \{f,g\} + \{f,c_q\}\{c_r,g\} - \{f,c_r\}\{c_q,g\} \\ &= \quad 0 \quad + \{f,\pi_q\}\{\pi_r,g\} - \{f,\pi_r\}\{\pi_q,g\} \\ &= -\frac{\partial f}{\partial q}\frac{\partial g}{\partial r} + \frac{\partial f}{\partial r}\frac{\partial g}{\partial q} \end{aligned}$$

This is the canonical Poisson bracket, with the role of momentum played by $r = q_1$.

Identifying $\pi = q_1$, all equations are Hamiltonian w.r.t. the canonical Poisson bracket and Hamiltonians

$$\begin{aligned} H_1(q,\pi) &= \frac{1}{2}|\pi|^2 + V_1(q) \qquad \text{and} \\ H_i(q,\pi) &= V_i(q,\pi) \quad \text{for } i \geq 2 \\ &= \pi_q q_i + \pi_r r_i - \mathcal{L}_i \qquad \text{on the constraint manifold.} \end{aligned}$$

Closedness and involutivity

Lemma

On solutions (identifying $\pi = p$) there holds

$$\mathrm{D}_i \mathcal{L}_j - \mathrm{D}_j \mathcal{L}_i = p_j q_i - p_i q_j = \{H_j, H_i\}^D.$$

Proof.

- Calculus of variations: for any smooth test function $\phi : \mathbb{R} \to \mathbb{R}$:

$$\int \delta \mathcal{L}_i(q, q_1, q_i) \phi(t_i) \, \mathrm{d}t_i = \int (p_i \delta q + p \delta q_i) \phi \, \mathrm{d}t_i$$

$$\Rightarrow \qquad \delta \mathcal{L}_i = p_i \delta q + p \delta q_i$$

Choosing $\delta = \mathrm{D}_j$, we obtain

$$\mathrm{D}_j \mathcal{L}_i = p_i q_j + p q_{ij}$$

Hence

$$\mathrm{D}_i \mathcal{L}_j - \mathrm{D}_j \mathcal{L}_i = p_j q_i - p_i q_j.$$

Closedness and involutivity

Lemma

On solutions (identifying $\pi = p$) there holds

$$\mathrm{D}_i \mathcal{L}_j - \mathrm{D}_j \mathcal{L}_i = p_j q_i - p_i q_j = \{H_j, H_i\}^D.$$

Proof.

- Calculus of variations:

$$\mathrm{D}_i \mathcal{L}_j - \mathrm{D}_j \mathcal{L}_i = p_j q_i - p_i q_j.$$

- Hamiltonian formalism:

$$\begin{aligned} \mathrm{D}_i \mathcal{L}_j - \mathrm{D}_j \mathcal{L}_i &= \{H_i, pq_j - H_j\}^D - \{H_j, pq_i - H_i\}^D \\ &= 2\{H_j, H_i\}^D + p_i q_j - p_j q_i. \end{aligned}$$

∎

Closedness and involutivity

Lemma

On solutions (identifying $\pi = p$) there holds

$$\mathrm{D}_i \mathcal{L}_j - \mathrm{D}_j \mathcal{L}_i = p_j q_i - p_i q_j = \{H_j, H_i\}^D.$$

Proof.

- Calculus of variations:

$$\mathrm{D}_i \mathcal{L}_j - \mathrm{D}_j \mathcal{L}_i = p_j q_i - p_i q_j.$$

- Hamiltonian formalism:

$$\mathrm{D}_i \mathcal{L}_j - \mathrm{D}_j \mathcal{L}_i = 2\{H_j, H_i\}^D + p_i q_j - p_j q_i.$$

∎

Theorem

The Hamiltonians are in involution with respect to the Dirac bracket if and only if $\mathrm{d}\mathcal{L} = 0$ on solutions.

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Hamiltonian structure of Lagrangian 2-form systems

Action integral

$$\int \mathcal{L}[u, u_t] \, \mathrm{d}x \wedge \mathrm{d}t,$$

where square brackets denote dependence on any number of space derivatives:

$$[u, u_t] = (u, u_t, u_x, u_{tx}, u_{xx}, u_{txx}, \ldots)$$

Assumption: the Lagrangian is linear in time-derivatives.

Then we can always find an equivalent Lagrangian of the form

$$\mathcal{L}[u, u_t] = p[u] u_t - V[u],$$

We introduce the constraint $c = \pi - p[u] = 0$ and take any Hamiltonian $H[u, \pi]$ satisfying

$$\begin{aligned} H[u, p[u]] &= p[u] u_t - \mathcal{L}[u, u_t] \\ &= V[u]. \end{aligned}$$

Dirac reduction in classical field theory

We have the constrained variational principle in phase space

$$\delta \int \left(H[u,\pi] - \pi u_t - \lambda(\pi - p[u]) \right) \mathrm{d}x \wedge \mathrm{d}t = 0,$$

yielding the equations

$$0 = \frac{\delta H}{\delta u} + \pi_t + \frac{\delta \lambda p}{\delta u}, \qquad 0 = \frac{\delta H}{\delta \pi} - u_t - \lambda, \qquad 0 = c.$$

Consider the Poisson bracket

$$\{ \textstyle\int f, \int g \} = \displaystyle\int \left(\frac{\delta_x f}{\delta \pi} \frac{\delta_x g}{\delta u} - \frac{\delta_x f}{\delta u} \frac{\delta_x g}{\delta \pi} \right) \mathrm{d}x$$

on the space of formal integrals (functions mod time derivatives).

The time-evolution of any functional $\int f(x, [u, \pi])$ is given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int f \, \mathrm{d}x &= \int \frac{\delta f}{\delta u} u_t + \frac{\delta f}{\delta \pi} \pi_t \, \mathrm{d}x \\ &= \{ \textstyle\int H, \int f \} - \{ \int \lambda c, \int f \}. \end{aligned}$$

Dirac reduction in classical field theory

The Poisson bracket

$$\{\textstyle\int f, \int g\} = \displaystyle\int \left(\frac{\delta_x f}{\delta \pi} \frac{\delta_x g}{\delta u} - \frac{\delta_x f}{\delta u} \frac{\delta_x g}{\delta \pi} \right) \mathrm{d}x.$$

does not satisfy the Leibniz rule (there is no multiplication on the space of functions mod x-derivatives). How to isolate λ from

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f \,\mathrm{d}x = \{\textstyle\int H, \int f\} - \{\int \lambda c, \int f\} \text{ ?}$$

Introduce the bracket

$$[f, g] = \sum_{k=0}^{\infty} \left(\mathrm{D}_x^k \left(\frac{\delta f}{\delta \pi_{x^k}} \right) \frac{\partial g}{\partial u_{x^k}} - \mathrm{D}_x^k \left(\frac{\delta f}{\delta u_{x^k}} \right) \frac{\partial g}{\partial \pi_{x^k}} \right)$$

It does satisfy the Leibniz rule in the second argument and

$$\left\{ \textstyle\int f \,\mathrm{d}x, \int g \,\mathrm{d}x \right\} = \displaystyle\int [f, g] \,\mathrm{d}x.$$

Dirac reduction in classical field theory

Let $\mathcal{M}$ be the operator defined by $\mathcal{M}\phi = [\phi c, c]$ for any smooth function $\phi(x)$. The Dirac brackets are given by

$$[f, g]^D = [f, g] - [f, c]\mathcal{M}^{-1}[g, c]$$

and

$$\left\{ \textstyle\int f, \int g \right\}^D = \int [f, g]^D \, \mathrm{d}x.$$

- $\{\cdot, \cdot\}^D$ is skew-symmetric and satisfies the Jacobi identity.
- For any smooth function $f[u, \pi]$ there holds

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f \, \mathrm{d}x = \left\{ \textstyle\int H, \int f \right\}^D.$$

- The constraint is a Casimir function for the Dirac bracket: for any smooth test function ϕ we have

$$\left\{ \textstyle\int \phi c, \int f \right\}^D = 0.$$

From pluri-Lagrangian to Hamiltonian systems

First row of coefficients of $\mathcal{L} = \sum_{i<j} \mathcal{L}_{ij} \, \mathrm{d}t_i \wedge \mathrm{d}t_j$:

$$\mathcal{L}_{1j} = p[u]u_j - h_j[u]$$

Impose the constraint $c = \pi - p[u] = 0$ and consider $H_{1j} = h_j[u]$.

The square bracket of the constraints is

$$[\phi c, c] = -\sum_I \phi_I \frac{\partial p[u]}{\partial u_I} + \frac{\delta \phi p[u]}{\delta u} =: \mathcal{E}_p \phi,$$

Hence the Dirac bracket is

$$\{\textstyle\int f, \int g\}^D = \{\int f, \int g\} - \displaystyle\int [f, c] \, \mathcal{E}_p^{-1} \, [g, c].$$

For functionals that do not depend on π, the it simplifies to

$$\{\textstyle\int f, \int g\}^D = \displaystyle\int \frac{\delta f}{\delta u} \, \mathcal{E}_p^{-1} \, \frac{\delta g}{\delta u}.$$

Example: potential KdV hierarchy

The pluri-Lagrangian structure for the KdV hierarchy has $p = \frac{1}{2} u_x$.

Hence $\mathcal{E}_p = \mathrm{D}_x$ and

$$\{\textstyle\int f, \int g\}^D = \displaystyle\int \frac{\delta f}{\delta u} D_x^{-1} \frac{\delta g}{\delta u}.$$

If f and g depend only on derivatives of u, this becomes the Gardner bracket

$$\{\textstyle\int f, \int g\}^D = \displaystyle\int \left(\mathrm{D}_x \frac{\delta f}{\delta u_x} \right) \frac{\delta g}{\delta u_x}.$$

The interpretation of the Gardner bracket for the KdV equation as a Dirac bracket was first given in:

[MacFarlane. Equations of Korteweg-De Vries type. I Lagrangian and Hamiltonian formalism. CERN, 1982]

Example: schwarzian KdV hierarchy

$$
\begin{aligned}
u_2 &= -\frac{3u_{11}^2}{2u_1} + u_{111} \\
u_3 &= -\frac{45u_{11}^4}{8u_1^3} + \frac{25u_{11}^2 u_{111}}{2u_1^2} - \frac{5u_{111}^2}{2u_1} - \frac{5u_{11}u_{1111}}{u_1} + u_{11111}, \qquad \dots
\end{aligned}
$$

has a pluri-Lagrangian structure with coefficients

$$
\mathcal{L}_{12} = \frac{u_2}{2u_1} - \frac{u_{11}^2}{2u_1^2}
$$

$$
\mathcal{L}_{13} = \frac{u_3}{2u_1} - \frac{3u_{11}^4}{8u_1^4} + \frac{u_{111}^2}{2u_1^2}
$$

$$
\begin{aligned}
\mathcal{L}_{23} = &-\frac{45u_{11}^6}{32u_1^6} + \frac{57u_{11}^4 u_{111}}{16u_1^5} - \frac{19u_{11}^2 u_{111}^2}{8u_1^4} + \frac{7u_{111}^3}{4u_1^3} - \frac{3u_{11}^3 u_{1111}}{4u_1^4} - \frac{3u_{11}u_{111}u_{1111}}{2u_1^3} + \frac{u_{1111}^2}{2u_1^2} \\
&+ \frac{3u_{11}^2 u_{11111}}{4u_1^3} - \frac{u_{111}u_{11111}}{2u_1^2} + \frac{u_{111}u_{112}}{u_1^2} - \frac{3u_{11}^3 u_{12}}{2u_1^4} + \frac{2u_{11}u_{111}u_{12}}{u_1^3} - \frac{u_{1111}u_{12}}{u_1^2} + \frac{u_{11}u_1}{u_1^2} \\
&- \frac{27u_{11}^4 u_2}{16u_1^5} + \frac{17u_{11}^2 u_{111}u_2}{4u_1^4} - \frac{7u_{111}^2 u_2}{4u_1^3} - \frac{3u_{11}u_{1111}u_2}{2u_1^3} + \frac{u_{11111}u_2}{2u_1^2} + \frac{u_{11}^2 u_3}{4u_1^3} - \frac{u_{111}u_3}{2u_1^2},
\end{aligned}
$$

. . .

Example: schwarzian KdV hierarchy

In this example we have $p = \frac{1}{2u_x}$, hence

$$\mathcal{E}_p = \frac{1}{u_x^2}\mathrm{D}_x - \frac{u_{xx}}{u_x^3}$$

and

$$\mathcal{E}_p^{-1} = u_x \mathrm{D}_x^{-1} u_x.$$

This nonlocal operator seems to be the simplest Hamiltonian operator for the SKdV equation.

The Hamilton functions for the first two flows are

$$H_2 = \frac{u_{11}^2}{2u_1^2} \qquad \text{and} \qquad H_3 = \frac{3u_{11}^4}{8u_1^4} - \frac{u_{111}^2}{2u_1^2}.$$

Closedness and involutivity

Proposition

On solutions of the multi-time Euler-Lagrange equations there holds

$$\{H_{1i}, H_{1j}\}^D = \int (p_i u_j - p_j u_i)\,\mathrm{d}x = \int (\mathrm{D}_j \mathcal{L}_{1i} - \mathrm{D}_i \mathcal{L}_{1j})\,\mathrm{d}x.$$

Since all quantities are defined modulo x-derivatives, we have

$$\int \mathrm{D}_1 \mathcal{L}_{ij}\,\mathrm{d}x \equiv 0,$$

hence

There holds $\{H_{1i}, H_{1j}\} = 0$ if and only if

$$\int (\mathrm{D}_1 \mathcal{L}_{ij} - \mathrm{D}_i \mathcal{L}_{1j} + \mathrm{D}_j \mathcal{L}_{1i})\,\mathrm{d}x = 0$$

on solutions of the multi-time Euler-Lagrange equations.

Conclusions

Context

- Integrability can be formulated in Lagrangian terms.
- Connections of the pluri-Lagrangian (Lagrangian multiform) theory to established notions of integrability are an active topic of research.

Progress

- A pluri-Lagrangian hierarchy also possesses a Hamiltonian structure (under mild conditions).

Open question

- Can we derive a bi-Hamiltonian structure from the pluri-Lagrangian formalism?

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From Hamiltonian to pluri-Lagrangian systems

Example: Kepler Problem. Poisson-commuting Hamiltonians

$$
\begin{aligned}
H_1(q,\pi) &= \frac{1}{2}|\pi|^2 + |q|^{-1}, && \text{energy} \\
H_2(q,\pi) &= (q \times \pi) \cdot e_z, && \text{3rd component of the ang. momentum} \\
H_3(q,\pi) &= |q \times \pi|^2, && \text{squared magnitude of the ang. momentum}
\end{aligned}
$$

where $q = (x,y,z)$ and e_z is the unit vector in the z-direction.

Lagrangian 1-form:

$$
\begin{aligned}
\mathcal{L}_1 &= \frac{1}{2}|q_1|^2 + |q|^{-1}, \\
\mathcal{L}_2 &= q_1 \cdot q_2 - (q \times q_1) \cdot e_z \\
\mathcal{L}_3 &= q_1 \cdot q_3 - |q \times q_1|^2
\end{aligned}
$$

From Hamiltonian to pluri-Lagrangian systems

The multi-time Euler-Lagrange equations are

$$\frac{\delta_1 \mathcal{L}_1}{\delta q} = 0 \quad \Rightarrow \quad q_{11} = \frac{q}{|q|^3},$$

the physical equations of motion,

$$\frac{\delta_2 \mathcal{L}_2}{\delta q_1} = 0 \quad \Rightarrow \quad q_2 = e_z \times q,$$
$$\frac{\delta_2 \mathcal{L}_2}{\delta q} = 0 \quad \Rightarrow \quad q_{12} = -q_1 \times e_z,$$

describing a rotation around the z-axis, and

$$\frac{\delta_3 \mathcal{L}_3}{\delta q_1} = 0 \quad \Rightarrow \quad q_3 = 2|q|^2 q_1 + 2(q \cdot q_1) q = 2(q \times q_1) \times q,$$
$$\frac{\delta_3 \mathcal{L}_3}{\delta q} = 0 \quad \Rightarrow \quad q_{13} = 2|q_1|^2 q - 2(q \cdot q_1) q_1,$$

describing a rotation around the angular momentum vector.

Alternative derivation

Note that the Hamiltonian operator D_x^{-1} can also be obtained without using Dirac reduction. Indeed, we can write the variational principle as

$$\int \frac{1}{2} u_t \mathrm{D}_x u - h[u] \, \mathrm{d}x \wedge \mathrm{d}t$$

which is the variational principle in phase space for the Hamiltonian equation $u_t = \mathrm{D}_x^{-1} \frac{\delta h}{\delta u}$. This approach works whenever we can write $p = Ju$ for some skew-adjoint operator J. This will not be the case in the next example.