How to find a pluri-Lagrangian structure for an integrable hierarchy?

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Hamiltonian Systems

Hamilton function

$$H : \mathbb{R}^{2N} \cong T^*Q \to \mathbb{R} : (q,p) \mapsto H(q,p)$$

determines dynamics:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

If $H = \frac{1}{2m} p^2 + U(q)$, then we find Newton's laws:

$$\dot{q} = \frac{1}{m} p \qquad \text{and} \qquad \dot{p} = -\nabla U(q)$$

Geometric interpretation:

- Phase space T^*Q with canonical symplectic 2-form ω
- flow along vector field X_H determined by $\iota_{X_H} \omega = \mathrm{d}H$
- the flows consists of symplectic maps and preserves H.

Poisson Brackets

Poisson bracket of two functionals on T^*Q:

$$\{f,g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

Dynamics of a Hamiltonian system:

$$\dot{q}_i = \{q_i, H\}, \qquad \dot{p}_i = \{p_i, H\}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} f(q,p) = \{f(q,p), H\}$$

Properties:

$$\begin{aligned} \text{anti-symmetry: } & \{f,g\} = -\{g,f\} \\ \text{bilinearity: } & \{f, g + \lambda h\} = \{f,g\} + \lambda\{f,h\} \\ \text{Leibniz property: } & \{f, gh\} = \{f,g\}h + g\{f,h\} \\ \text{Jacobi identity: } & \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \end{aligned}$$

Liouville-Arnold integrability

What if $H(p,q) = H(p)$?

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} = 0 \qquad \text{and} \qquad \dot{q}_j = \frac{\partial H}{\partial p_j} = \omega_j(p) = \mathrm{const}$$

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A Hamiltonian system with Hamilton function $H : \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville-Arnold integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \dots H_N$ such that $\{H_i, H_j\} = 0$.

- each H_i is a conserved quantity for all flows.
- the dynamics is confined to a leaf of the foliation $\{H_i = \mathrm{const}\}$.
- the flows commute.
- There exists a symplectic change of variables $(p,q) \mapsto (\bar{p}, \bar{q})$ such that

  $$H(p,q) = \bar{H}_i(\bar{p})$$

  Liouville-Arnold integrable systems evolve linearly in these variables!

  $(\bar{p}, \bar{q})$ are called action-angle variables.

Variational analogue of $\{H_i, H_j\} = 0$

Integrable systems come in families: finite (for ODEs) or infinite (for PDEs) hierarchies of commuting equations:

$$\frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\mathrm{d}}{\mathrm{d}t_j} = \frac{\mathrm{d}}{\mathrm{d}t_j}\frac{\mathrm{d}}{\mathrm{d}t_i} \qquad \text{for time variables } t_1, t_2, \ldots$$

On the Hamiltonian side, integrability is characterized by $\{H_i, H_j\} = 0$.

What about the Lagrangian side?

Variational analogue of $\{H_i, H_j\} = 0$

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On the Hamiltonian side, integrability is characterized by $\{H_i, H_j\} = 0$.

What about the Lagrangian side?

Pluri-Lagrangian principle ($d = 1$)

Combine the Lagrange functions $L_i[u]$ into a Lagrangian 1-form

$$\mathcal{L}[u] = \sum_i L_i[u]\,\mathrm{d}t_i.$$

Look for dynamical variables $u(t_1, \ldots, t_N)$ such that the action

$$S_\Gamma = \int_\Gamma \mathcal{L}[u]$$

is critical w.r.t. variations of u, simultaneously over every curve Γ in multi-time $\mathbb{R}^N$

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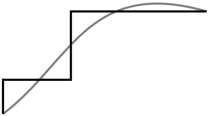
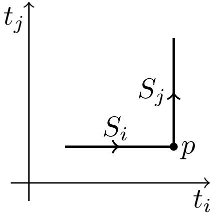
Multi-time Euler-Lagrange equations

Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i[u] \, \mathrm{d}t_i$

Lemma

If the action $\int_S \mathcal{L}$ is critical on all stepped curves S in $\mathbb{R}^N$, then it is critical on all smooth curves.

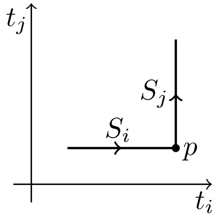




Variations are local, so it is sufficient to look at a general L-shaped curve $S = S_i \cup S_j$.

Multi-time Euler-Lagrange equations

$$\begin{aligned}
\delta \int_{S_i} L_i \, \mathrm{d}t_i &= \int_{S_i} \sum_I \frac{\partial L_i}{\partial u_I} \delta u_I \, \mathrm{d}t_i \\
&= \int_{S_i} \sum_{I \not\ni t_i} \sum_{\alpha=0}^{\infty} \frac{\partial L_i}{\partial u_{It_i^\alpha}} \delta u_{It_i^\alpha} \, \mathrm{d}t_i \\
&= \int_{S_i} \sum_{I \not\ni t_i} \frac{\delta_i L_i}{\delta u_I} \delta u_I \, \mathrm{d}t_i + \sum_I \frac{\delta_i L_i}{\delta u_{It_i}} \delta u_I \bigg|_p ,
\end{aligned}$$



where I denotes a multi-index, and

$$\frac{\delta_i L_i}{\delta u_I} = \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial L_i}{\partial u_{It_i^\alpha}} = \frac{\partial L_i}{\partial u_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial u_{It_i}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial L_i}{\partial u_{It_i^2}} - \dots$$

Multi-time Euler-Lagrange equations for curves, $\mathcal{L} = \sum_i L_i[u] \, \mathrm{d}t_i$

$$\frac{\delta_i L_i}{\delta u_I} = 0 \qquad \forall I \not\ni t_i \qquad \text{and} \qquad \frac{\delta_i L_i}{\delta u_{It_i}} = \frac{\delta_j L_j}{\delta u_{It_j}} \qquad \forall I,$$

Example: Kepler Problem

The classical Lagrangian of a particle in the gravitational potential

$$L_1[q] = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

can be combined with

$$L_2[q] = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot e,$$

into a pluri-Lagrangian 1-form $L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$ and consider $q = q(t_1, t_2)$.

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Multi-time Euler-Lagrange equations:

$$\frac{\delta_1 L_1}{\delta q} = 0 \quad \Rightarrow \quad q_{t_1 t_1} = -\frac{q}{|q|^3} \qquad \text{(Keplerian motion)}$$

$$\frac{\delta_2 L_2}{\delta q} = 0 \quad \Rightarrow \quad q_{t_1 t_2} = e \times q_{t_1}$$

$$\frac{\delta_2 L_2}{\delta q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_2} = e \times q \qquad \text{(Rotation)}$$

$$\frac{\delta_1 L_1}{\delta q_{t_1}} = \frac{\delta_2 L_2}{\delta q_{t_2}} \quad \Rightarrow \quad q_{t_1} = q_{t_1}$$

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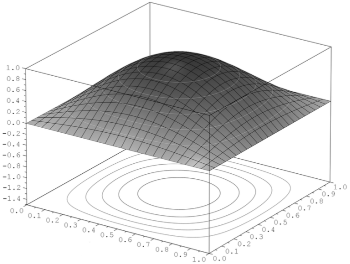
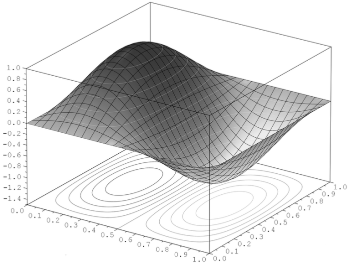
Pluri-Lagrangian principle ($d = 2$, continuous)

Given a 2-form

$$\mathcal{L} = \sum_{i,j} L_{ij}[u] \,\mathrm{d}t_i \wedge \mathrm{d}t_j,$$

find a field $u : \mathbb{R}^N \to \mathbb{C}$, such that $\displaystyle\int_\Gamma \mathcal{L}$ is critical on all smooth 2-surfaces Γ in multi-time $\mathbb{R}^N$, w.r.t. variations of u.



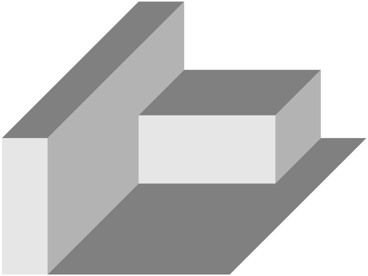


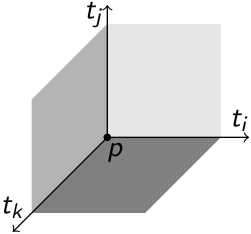
Example: KdV hierarchy, where $t_1 = x$ is the shared space coordinate, t_i time for i-th flow. (Details to follow.)

Multi-time EL equations

Consider a Lagrangian 2-form $\mathcal{L} = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$.

Every smooth surface can be approximated arbitrarily well by stepped surfaces. Hence it is sufficient to require criticality on stepped surfaces, or just on their elementary corners.





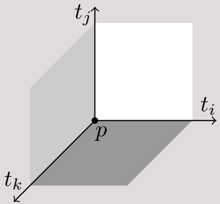
Multi-time EL equations

for $\mathcal{L} = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{It_k t_i}} = 0 \qquad \forall I.$$



Where

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial u_{It_i^\alpha t_j^\beta}}$$

Example: Potential KdV hierarchy

$u_{t_2} = Q_2 = u_{xxx} + 3u_x^2,$

$u_{t_3} = Q_3 = u_{xxxxx} + 10u_x u_{xxx} + 5u_{xx}^2 + 10u_x^3,$

where we identify $t_1 = x$.

The differentiated equations $u_{xt_i} = \frac{\mathrm{d}}{\mathrm{d}x} Q_i$ are Lagrangian with

$$L_{12} = \frac{1}{2} u_x u_{t_2} - \frac{1}{2} u_x u_{xxx} - u_x^3,$$

$$L_{13} = \frac{1}{2} u_x u_{t_3} - u_x u_{xxxxx} - 2u_{xx} u_{xxxx} - \frac{3}{2} u_{xxx}^2 + 5u_x^2 u_{xxx} + 5u_x u_{xx}^2 + \frac{5}{2} u_x^4.$$

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A suitable coefficient L_{23} of

$$\mathcal{L} = L_{12} \,\mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13} \,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23} \,\mathrm{d}t_2 \wedge \mathrm{d}t_3$$

can be found (nontrivial task!) in the form

$$L_{23} = \frac{1}{2}(u_{t_2} Q_3 - u_{t_3} Q_2) + p_{23}.$$

Example: Potential KdV hierarchy

- The equations $\dfrac{\delta_{12} L_{12}}{\delta u} = 0$ and $\dfrac{\delta_{13} L_{13}}{\delta u} = 0$ yield

$$u_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x} Q_2 \qquad \text{and} \qquad u_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x} Q_3.$$

- The equations $\dfrac{\delta_{12} L_{12}}{\delta u_x} = \dfrac{\delta_{32} L_{32}}{\delta u_{t_3}}$ and $\dfrac{\delta_{13} L_{13}}{\delta u_x} = \dfrac{\delta_{23} L_{23}}{\delta u_{t_2}}$ yield

$$u_{t_2} = Q_2 \qquad \text{and} \qquad u_{t_3} = Q_3,$$

  the evolutionary equations!

- All other multi-time EL equations are corollaries of these.

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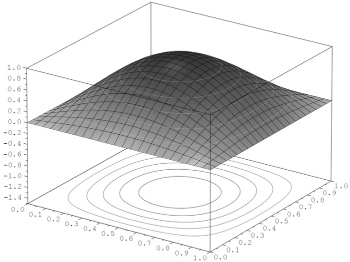
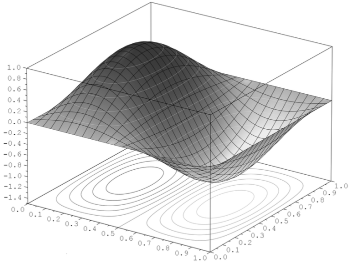
Closedness of the Lagrangian form

One could require additionaly that $\mathcal{L}$ is closed on solutions
$\hookrightarrow$ "Lagrangian multiform systems".

Then the action is not just critical on every curve/surface, but also takes the same value on every curve/surface.





We do not take this as part of the definition, because one can show

Proposition

$\mathrm{d}\mathcal{L}$ is constant on the set of solutions.

Closedness relates to other notions of integrability

If $\mathrm{d}\left(\sum_i L_i \,\mathrm{d}t_i \right) = 0$, then $\dfrac{\mathrm{d}L_k}{\mathrm{d}t_j} = \dfrac{\mathrm{d}L_j}{\mathrm{d}t_k}$

Variational symmetries

t_j-flow changes L_k by a t_k-derivative.
$\Rightarrow$ Individual flows are variational symmetries of each other:

If $\mathrm{d}\left(\sum_{i,j} L_{ij} \,\mathrm{d}t_i \wedge \mathrm{d}t_j \right) = 0$, then $\dfrac{\mathrm{d}L_{ij}}{\mathrm{d}t_k} = \dfrac{\mathrm{d}L_{ik}}{\mathrm{d}t_j} - \dfrac{\mathrm{d}L_{jk}}{\mathrm{d}t_i}$

Variational symmetries

t_k-flow changes L_{ij} by a divergence in (t_i, t_j).
$\Rightarrow$ Individual flows are variational symmetries of each other.

Idea: use variational symmetries to construct a pluri-Lagrangian structure

Variational Symmetries and pluri-Lagrangian structures

1-forms [Petrera, Suris. 2017]

Given a mechanical Lagrangian L_1 and a number of variational symmetries, we can construct coefficents L_i such that the pluri-Lagrangian 1-form

$$\sum_i L_i \, \mathrm{d}t_i$$

describes the mechanical system coupled with its variational symmetries.

A similar result holds for Lagrangian 2-forms [Petrera, V. In preparation].

Given Lagrangians L_{1j} corresponding to the individual PDEs of a hierarchy, such that each PDE is a variational symmetry for the other Lagrangians, we can find also L_{ij} with $i, j > 1$ such that

$$\sum_{i,j} L_{ij} \, \mathrm{d}t_i \wedge \mathrm{d}t_j.$$

is a pluri-Lagrangian structure for the hierarchy.

Preliminaries

We think of the field $u(t_1, t_2)$ as a section of the bundle $\mathbb{R}^2(t_1, t_2) \times \mathbb{R}(u)$.
The derivatives of u live in the infinite jet bundle
$\mathbb{R}^2(t_1, t_2) \times \mathcal{J}^\infty(u, u_{t_1}, u_{t_2}, u_{t_1 t_1}, u_{t_1 t_2}, u_{t_2 t_2}, \ldots)$.

- A vertical generalized vector field on $\mathbb{R}^2(t_1, t_2) \times \mathbb{R}(u)$ is a vector field of the form $Q\partial_u$, where Q depends on u and its derivatives.
- The prolongation of $Q\partial_u$ is a vector field on the infinite jet $\mathcal{J}^\infty$ defined as

$$\mathrm{pr}(Q\partial_u) = \sum_{I \in \mathbb{N}^2} (\mathrm{D}_I Q) \frac{\partial}{\partial u_I}.$$

(Motivation: describe the action of $Q\partial_u$ on a function of the jet $\mathcal{J}^\infty$.)

Preliminaries

- $Q\partial_u$ is a variational symmetry of $L : \mathcal{J}^\infty \to \mathbb{R}$ if its prolongation satisfies
$$\mathrm{pr}(Q\partial_u)L = \mathrm{D}_1 F_1 + \mathrm{D}_2 F_2$$
for some $F_1, F_2 : \mathcal{J}^\infty \to \mathbb{R}$, where D_i is the total derivative w.r.t. t_i.
- A conservation law for $L : \mathcal{J}^\infty \to \mathbb{R}$ is a triple $J_1, J_2, Q : \mathcal{J}^\infty \to \mathbb{R}$ that satisfy
$$\mathrm{D}_1 J_1 + \mathrm{D}_2 J_2 = -Q \frac{\delta L}{\delta u}.$$
$J = (J_1, J_2)$ is called the conserved current Q is called the characteristic of the conservation law.

On solutions: $\mathrm{div} J = 0$.

Noether's theorem for 2-dimensional PDEs

Theorem

Let $Q\partial_u$ be a variational symmetry of L, i.e. $\operatorname{pr}(Q\partial_u)L = \mathrm{D}_1 F_1 + \mathrm{D}_2 F_2$. Then

$$J_1 = \sum_{I \not\ni t_2} \left((\mathrm{D}_I Q) \frac{\delta L}{\delta u_{I t_1}} \right) + \frac{1}{2} \sum_I \mathrm{D}_2 \left((\mathrm{D}_I Q) \frac{\delta L}{\delta u_{I t_1 t_2}} \right) - F_1,$$

$$J_2 = \sum_{I \not\ni t_1} \left((\mathrm{D}_I Q) \frac{\delta L}{\delta u_{I t_2}} \right) + \frac{1}{2} \sum_I \mathrm{D}_1 \left((\mathrm{D}_I Q) \frac{\delta L}{\delta u_{I t_1 t_2}} \right) - F_2$$

are the components of the conserved current of a conservation law.

Proof. Integration by parts of $\operatorname{pr}(Q\partial_u)L = \sum_I (\mathrm{D}_I Q) \frac{\partial L}{\partial u_I}$ to get

$$\operatorname{pr}(Q\partial_u)L = Q \frac{\delta L}{\delta u} + \mathrm{D}_1(\cdots) + \mathrm{D}_2(\cdots)$$

hence

$$-Q \frac{\delta L}{\delta u} = \mathrm{D}_1(\cdots - F_1) + \mathrm{D}_2(\cdots - F_1).$$

∎

From variational symmetries to a pluri-Lagrangian 2-form

Consider a hierarchy of PDEs

$$u_i = Q_i(u_1, u_{11}, \ldots) \qquad i = 2, \ldots, N,$$

with their Lagrangians

$$L_{1i} = p(u, u_1, u_{11}, \ldots) u_i - h(u, u_1, u_{11}, \ldots)$$

Assume that the prolonged vector fields $\mathrm{pr}(Q_i \partial_u)$, commute and are variational symmetries of the L_{1j}:

$$\mathrm{pr}(Q_i \partial_u) L_{1j} = \mathrm{D}_1 A_{ij} + \mathrm{D}_j B_{ij}$$

Lemma

There exist F_{ij} such that

$$\mathrm{D}_1 F_{ij} = \mathrm{D}_i L_{1j} - \mathrm{D}_j L_{1i}$$

on solutions of the hierarchy

Sketch of proof. Show that $\int_{-\infty}^{\infty} \mathrm{D}_i L_{1j} - \mathrm{D}_j L_{1i} \, \mathrm{d}t_1 = 0$. ∎

From variational symmetries to a pluri-Lagrangian 2-form

Lemma

There exist F_{ij} such that $\mathrm{D}_1 F_{ij} = \mathrm{D}_i L_{1j} - \mathrm{D}_j L_{1i}$ on solutions of the hierarchy

Theorem

For $i, j > 1$, let

$$L_{ij}[u] = \sum_{\alpha \geq 0} \frac{\delta_{1j} L_{1j}}{\delta u_{t_1^{\alpha+1}}} \mathrm{D}_1^\alpha (u_i - Q_i) - \sum_{\alpha \geq 0} \frac{\delta_{1i} L_{1i}}{\delta u_{t_1^{\alpha+1}}} \mathrm{D}_1^\alpha (u_j - Q_j) + F_{ij}(u, u_1, u_{11}, \ldots)$$

Then every solution of the hierarchy

$$u_i = Q_i(u_1, u_{11}, \ldots) \qquad i = 2, \ldots, N,$$

is a critical point of

$$\mathcal{L}[u] = \sum_{i<j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j.$$

From variational symmetries to a pluri-Lagrangian 2-form

Theorem

$$L_{ij}[u] = \sum_{\alpha \geq 0} \frac{\delta_{1j} L_{1j}}{\delta u_{t_1^{\alpha+1}}} \mathrm{D}_1^\alpha (u_i - Q_i) - \sum_{\alpha \geq 0} \frac{\delta_{1i} L_{1i}}{\delta u_{t_1^{\alpha+1}}} \mathrm{D}_1^\alpha (u_j - Q_j) + F_{ij}(u, u_1, u_{11}, \ldots)$$

Sketch of proof. Show that $\mathrm{d}\mathcal{L}$ attains a double zero on solutions of the hierarchy.

Then, on solutions:

$$\mathrm{d}\left(\mathfrak{D}_{\mathrm{pr}(v\partial_u)}\mathcal{L}\right) = \mathfrak{D}_{\mathrm{pr}(v\partial_u)}\mathrm{d}\mathcal{L} = 0,$$

where $\mathfrak{D}$ is the Lie derivative. Hence locally there exists a 1-form Θ such that

$$\mathfrak{D}_{\mathrm{pr}(v\partial_u)}\mathcal{L} = \mathrm{d}\Theta$$

so for variations $v\partial_u$ that vanish on $\partial\Gamma$ we find

$$\int_\Gamma \mathfrak{D}_{\mathrm{pr}(v\partial_u)}\mathcal{L} = 0,$$

■

Example: Potential KdV hierarchy

$$
\begin{aligned}
u_2 &= Q_2 = 3u_1^2 + u_{111}, \\
u_3 &= Q_3 = 10u_1^3 + 5u_{11}^2 + 10u_1 u_{111} + u_{11111}.
\end{aligned}
$$

The corresponding Lagrangians are

$$
\begin{aligned}
L_{12} &= \frac{1}{2} u_1 u_2 - u_1^3 - \frac{1}{2} u_1 u_{111}, \\
L_{13} &= \frac{1}{2} u_1 u_3 - \frac{5}{2} u_1^4 + 5 u_1 u_{11}^2 - \frac{1}{2} u_{111}^2,
\end{aligned}
$$

On solutions of the evolutionary equations, there holds

$$
\begin{aligned}
\mathrm{D}_2 L_{13} - \mathrm{D}_3 L_{12} &= -10u_1^3 u_{12} + 10 u_1 u_{11} u_{112} + 5u_{11}^2 u_{12} + 3u_1^2 u_{13} \\
&\quad - u u_{111} u_{1112} + \tfrac{1}{2} u_1 u_{1113} + \tfrac{1}{2} u_{111} u_{13} - \tfrac{1}{2} u_{13} u_2 + \tfrac{1}{2} u_{12} u_3 \\
&= 15 u_1^4 u_{11} + 135 u_1 u_{11}^3 + 210 u_1^2 u_{11} u_{111} + 25 u_1^3 u_{1111} \\
&\quad - 18 u_{11} u_{111}^2 + \tfrac{15}{2} u_{11}^2 u_{1111} + 34 u_1 u_{111} u_{1111} \\
&\quad + 33 u_1 u_{11} u_{11111} + \tfrac{13}{2} u_1^2 u_{111111} + \tfrac{1}{2} u_{1111} u_{11111} \\
&\quad - u_{111} u_{111111} + \tfrac{1}{2} u_1 u_{11111111}.
\end{aligned}
$$

Example: Potential KdV hierarchy

Integrating

$$\begin{aligned}\mathrm{D}_2 L_{13} - \mathrm{D}_3 L_{12} = {} & 15 u_1^4 u_{11} + 135 u_1 u_{11}^3 + 210 u_1^2 u_{11} u_{111} + 25 u_1^3 u_{1111} \\ & - 18 u_{11} u_{111}^2 + \tfrac{15}{2} u_{11}^2 u_{1111} + 34 u_1 u_{111} u_{1111} \\ & + 33 u_1 u_{11} u_{11111} + \tfrac{13}{2} u_1^2 u_{111111} + \tfrac{1}{2} u_{1111} u_{11111} \\ & - u_{111} u_{111111} + \tfrac{1}{2} u_1 u_{11111111}\end{aligned}$$

gives

$$\begin{aligned}F_{23} = {} & 3 u_1^5 + \tfrac{135}{2} u_1^2 u_{11}^2 + 25 u_1^3 u_{111} - \tfrac{25}{2} u_{11}^2 u_{111} + 7 u_1 u_{111}^2 + 20 u_1 u_{11} u_{1111} \\ & + \tfrac{13}{2} u_1^2 u_{11111} + \tfrac{1}{2} u_{1111}^2 - \tfrac{1}{2} u_{111} u_{11111} - \tfrac{1}{2} u_{11} u_{111111} + \tfrac{1}{2} u_1 u_{1111111}.\end{aligned}$$

Now we can calculate the coefficient L_{23} as

$$L_{23} = \sum_{\alpha \geq 0} \frac{\delta_{13} L_{13}}{\delta u_{t_1^{\alpha+1}}} \mathrm{D}_1^\alpha (u_2 - Q_2) - \sum_{\alpha \geq 0} \frac{\delta_{12} L_{12}}{\delta u_{t_1^{\alpha+1}}} \mathrm{D}_1^\alpha (u_3 - Q_3) + F_{23}$$

Example: Potential KdV hierarchy

The first two equations of the potential KdV hierarchy have a pluri-Lagrangian structure

$$\mathcal{L} = L_{12}\,\mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}\,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}\,\mathrm{d}t_2 \wedge \mathrm{d}t_3$$

with

$$\begin{aligned}
L_{12} &= \tfrac{1}{2}u_1 u_2 - u_1^3 - \tfrac{1}{2}u_1 u_{111}, \\
L_{13} &= \tfrac{1}{2}u_1 u_3 - \tfrac{5}{2}u_1^4 + 5u_1 u_{11}^2 - \tfrac{1}{2}u_{111}^2, \\
L_{23} &= 3u_1^5 - \tfrac{15}{2}u_1^2 u_{11}^2 + 10u_1^3 u_{111} - 5u_1^3 u_2 + \tfrac{7}{2}u_{11}^2 u_{111} + 3u_1 u_{111}^2 \\
&\quad - 6u_1 u_{11} u_{1111} + \tfrac{3}{2}u_1^2 u_{11111} + 10u_1 u_{11} u_{12} - \tfrac{5}{2}u_{11}^2 u_2 - 5u_1 u_{111} u_2 \\
&\quad + \tfrac{3}{2}u_1^2 u_3 - \tfrac{1}{2}u_{1111}^2 + \tfrac{1}{2}u_{111} u_{11111} - u_{111} u_{112} + \tfrac{1}{2}u_1 u_{113} \\
&\quad + u_{1111} u_{12} - \tfrac{1}{2}u_{11} u_{13} - \tfrac{1}{2}u_{11111} u_2 + \tfrac{1}{2}u_{111} u_3.
\end{aligned}$$

For any n we can construct a Lagrangian 2-form in n dimensions, describing $n-1$ equations of the hierarchy.

Further examples

Nonlinear Schrödinger hierarchy

$$
\begin{aligned}
u_2 &= Q_2 = i u_{11} - 2i|u|^2 u, & \bar{u}_2 &= \bar{Q}_2 = -i\bar{u}_{11} + 2i|\bar{u}|^2\bar{u}, \\
u_3 &= Q_2 = u_{111} - 6|u|^2 u_1, & \bar{u}_3 &= \bar{Q}_3 = \bar{u}_{111} - 6|\bar{u}|^2\bar{u}_1, \\
&\dots & &\dots
\end{aligned}
$$

with Lagrangians

$$
\begin{aligned}
L_{12}[u] &= \frac{i}{2}\left(u_2\bar{u} - u\bar{u}_2\right) - |u_1|^2 - |u|^4, \\
L_{13}[u] &= \frac{i}{2}\left(u_3\bar{u} - u\bar{u}_3\right) + \frac{i}{2}\left(u_{11}\bar{u}_1 - u_1\bar{u}_{11}\right) + \frac{3i}{2}|u|^2\left(u_1\bar{u} - u\bar{u}_1\right)
\end{aligned}
$$

Slight generalization needed to deal with 2-component Lagrangians:

$$
\begin{aligned}
L_{23} = &\sum_{\alpha\geq 0} \frac{\delta_{13}L_{13}}{\delta u_{t_1^{\alpha+1}}} \mathrm{D}_1^\alpha(u_2 - Q_2) - \sum_{\alpha\geq 0} \frac{\delta_{12}L_{12}}{\delta u_{t_1^{\alpha+1}}} \mathrm{D}_1^\alpha(u_3 - Q_3) \\
&+ \sum_{\alpha\geq 0} \frac{\delta_{13}L_{13}}{\delta \bar{u}_{t_1^{\alpha+1}}} \mathrm{D}_1^\alpha(\bar{u}_2 - \bar{Q}_2) - \sum_{\alpha\geq 0} \frac{\delta_{12}L_{12}}{\delta \bar{u}_{t_1^{\alpha+1}}} \mathrm{D}_1^\alpha(\bar{u}_3 - \bar{Q}_3) + F_{23}
\end{aligned}
$$

Further examples

Sine-Gordon and the potential modified KdV hierarchy

$$u_{12} = \sin u$$

$$u_3 = Q_3 = u_{111} + \frac{1}{2} u_1^3,$$

$$\dots$$

with Lagrangians

$$L_{12}[u] = \frac{1}{2} u_1 u_2 - \cos u,$$

$$L_{13}[u] = \frac{1}{2} u_1 u_3 - \frac{1}{8} u_1^4 + \frac{1}{2} u_{11}^2,$$

$$\dots$$

Further examples

Sine-Gordon and the potential modified KdV hierarchy

$$u_{12} = \mathrm{D}_1 Q_2 = \sin u$$
$$u_3 = Q_3 = u_{111} + \frac{1}{2} u_1^3,$$
$$\dots$$

with Lagrangians

$$L_{12}[u] = \frac{1}{2} u_1 u_2 - \cos u,$$
$$L_{13}[u] = \frac{1}{2} u_1 u_3 - \frac{1}{8} u_1^4 + \frac{1}{2} u_{11}^2,$$
$$\dots$$

Slight generalization needed because Sine-Gordon is not evolutionary:

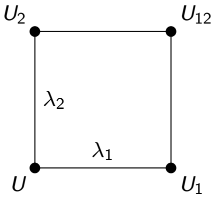
$$L_{23} = \sum_{\alpha \geq 1} \frac{\delta_{13} L_{13}}{\delta u_{t_1^{\alpha+1}}} \mathrm{D}_1^\alpha (u_2 - Q_2) - \sum_{\alpha \geq 0} \frac{\delta_{12} L_{12}}{\delta u_{t_1^{\alpha+1}}} \mathrm{D}_1^\alpha (u_3 - Q_3) + F_{23}$$

Table of Contents

Quad equations

$$\mathcal{Q}(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2) = 0$$

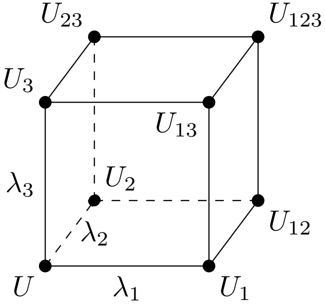
Subscripts of U denote lattice shifts, λ_1, λ_2 are parameters.
Invariant under symmetries of the square, affine in each of U, U_1, U_2, U_{12}.



Integrability for systems quad equations:
Multi-dimensional consistency of

$$\mathcal{Q}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j) = 0,$$

i.e. the thrunderee ways of calculating U_{123} give the same result.



Classification (under some extra assumptions) by Adler, Bobenko and Suris (ABS List).

Example: lattice potential KdV:
$(U - U_{12})(U_1 - U_2) - \lambda_1 + \lambda_2 = 0$

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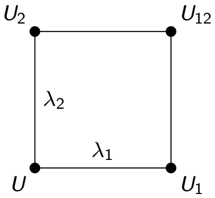

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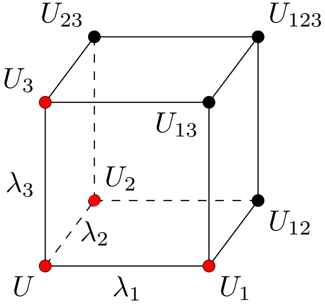

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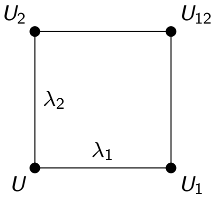

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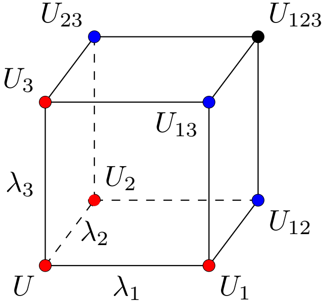


Classification (under some extra assumptions) by Adler, Bobenko and Suris (ABS List).

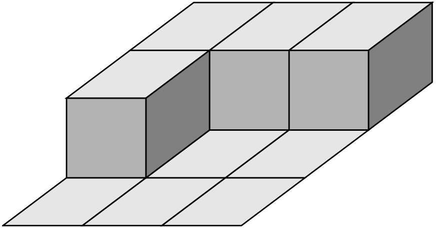
Example: lattice potential KdV:
$(U - U_{12})(U_1 - U_2) - \lambda_1 + \lambda_2 = 0$

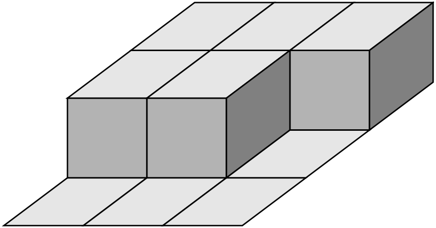
Pluri-Lagrangian structure for quad equations

For some discrete 2-form

$$\mathcal{L}(\square_{ij}) = \mathcal{L}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j),$$

the action $\displaystyle\sum_{\square\in\Gamma} \mathcal{L}(\square)$ is critical on all 2-surfaces Γ in $\mathbb{N}^N$ simultaneously.





To derive Euler-Lagrange equations: vary U at each point individually.

$\hookrightarrow$ It is sufficient to consider corners of an elementary cube.

[Lobb, Nijhoff. 2009]

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Miwa shifts

Naive continuum limits of quad equations do not lead to integrable PDEs.

Continuum limit of an integrable difference equation

Skew embedding of the mesh $\mathbb{Z}^N$ into multi-time $\mathbb{R}^N$
Discrete Q is a sampling of the continuous q:

$$Q = Q(\mathbf{n}) = q(t_1, t_2, \ldots, t_N),$$

$$Q_i = Q(\mathbf{n} + \mathfrak{e}_i) = q\left(t_1 - 2\lambda_i, t_2 + 2\frac{\lambda_i^2}{2}, \ldots, t_N + 2(-1)^N \frac{\lambda_i^N}{N} \right)$$

[Miwa. On Hirota's difference equations. Proceedings of the Japan Academy A. 1982]

Write quad equation in terms of q and expand in λ_1.

Continuum limit of H1 (lattice potential KdV)

$$\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+U_{1,2}-U\right)\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}+U_2-U_1\right)=\frac{1}{\lambda_2^2}-\frac{1}{\lambda_1^2} \qquad \text{(lpKdV)}$$

This is a well-chosen representative of H1 out of many equivalent forms.
Often one finds it written as $(X-X_{12})(X_2-X_1)=\alpha_1-\alpha_2$

Miwa shifts

$$U=U(\mathbf{n})=u(t_1,t_2,\dots,t_n),$$

$$U_i=U(\mathbf{n}+\mathfrak{e}_i)=u\left(t_1-2\lambda_i,t_2+2\frac{\lambda_i^2}{2},\dots,t_n+2(-1)^n\frac{\lambda_i^N}{N}\right)$$

Plug into (lpKdV) and expand in λ_1, λ_2.
In leading order everything cancels due to very specific form of quad eqn.

Generically we would have an ODE in t_1, e.g.

$$(X-X_{12})(X_2-X_1)=\lambda_1^2-\lambda_2^2 \;\Rightarrow\; 4(\lambda_1+\lambda_2)x_{t_1}(\lambda_1-\lambda_2)x_{t_1}=\lambda_1^2-\lambda_2^2$$
$$\Rightarrow\; x_{t_1}^2=\tfrac{1}{4}$$

Continuum limit of H1 (lattice potential KdV)

Series expansion

$$\text{Quad Equation} \quad \to \quad \sum_{i,j} \frac{4}{ij} f_{i,j}[u] \, \lambda_1^i \lambda_2^j = 0,$$

where $f_{j,i} = -f_{i,j}$ and the factor $\dfrac{4}{ij}$ is chosen to normalize the $f_{0,j}$.

First row of coefficients:

$$f_{0,1} = -u_{t_2},$$

$$f_{0,2} = -3u_{t_1}^2 - u_{t_1 t_1 t_1} - \frac{3}{2} u_{t_1 t_2} + u_{t_3},$$

$$f_{0,3} = 8u_{t_1} u_{t_1 t_1} + 4u_{t_1} u_{t_2} + \frac{4}{3} u_{t_1 t_1 t_1 t_1} - \frac{4}{3} u_{t_1 t_3} - u_{t_2 t_2} - u_{t_4},$$

$$\begin{aligned} f_{0,4} = {} & -5u_{t_1 t_1}^2 - \frac{20}{3} u_{t_1} u_{t_1 t_1 t_1} + 10 u_{t_1} u_{t_1 t_2} + 5 u_{t_1 t_1} u_{t_2} - \frac{5}{4} u_{t_2}^2 - \frac{10}{3} u_{t_1} u_{t_3} \\ & - u_{t_1 t_1 t_1 t_1 t_1} + \frac{5}{3} u_{t_1 t_1 t_1 t_2} + \frac{5}{4} u_{t_1 t_2 t_2} - \frac{5}{4} u_{t_1 t_4} - \frac{5}{3} u_{t_2 t_3} + u_{t_5}, \end{aligned}$$

Continuum limit of H1 (lattice potential KdV)

Setting each f_{ij} equal to zero, we find

$$\begin{aligned} u_{t_2} &= 0, \\ u_{t_3} &= 3u_{t_1}^2 + u_{t_1 t_1 t_1} \\ u_{t_4} &= 0, \\ u_{t_5} &= 10u_{t_1}^3 + 5u_{t_1 t_1}^2 + 10 u_{t_1} u_{t_1 t_1 t_1} + u_{t_1 t_1 t_1 t_1 t_1}, \\ &\vdots \end{aligned}$$

$\hookrightarrow$ pKdV hierarchy

Whole hierarchy from single quad equation

using Miwa correspondence

$$U = U(\mathbf{n}) = u(t_1, t_2, \ldots, t_n),$$

$$U_i = U(\mathbf{n} + \mathfrak{e}_i) = u\left(t_1 - 2\lambda_i, t_2 + 2\frac{\lambda_i^2}{2}, \ldots, t_n + 2(-1)^n \frac{\lambda_i^N}{N} \right)$$

Continuum limit of the Lagrangian

- Using Miwa correspondence:

$$\text{Discrete } L \quad \to \quad \text{Power series } \mathcal{L}_{\mathrm{disc}}[u(\mathbf{t})]$$

  Action for $\mathcal{L}_{\mathrm{disc}}[u(\mathbf{t})]$ is still a sum.

- Two applications of the Euler-Maclaurin formula:

$$\mathcal{L}_{\mathrm{Miwa}}([u], \lambda_1, \lambda_2) = \sum_{i,j=0}^{\infty} \frac{B_i B_j}{i!j!} \partial_1^i \partial_2^j \mathcal{L}_{\mathrm{disc}}([u], \lambda_1, \lambda_2).$$

  where the differential operators are $\quad \partial_k = \sum_{j=1}^{N} (-1)^{j+1} \frac{2\lambda_k^j}{j} \frac{\mathrm{d}}{\mathrm{d}t_j}$

- Then there holds $L_{\mathrm{disc}}(\square) = \int_{\blacksquare} \mathcal{L}_{\mathrm{Miwa}}([u(\mathbf{t})], \lambda_1, \lambda_2)\, \eta_1 \wedge \eta_2,$

  where η_1 and η_2 are the 1-forms dual to the Miwa shifts.

  This suggests the Lagrangian 2-form

$$\sum_{1 \le i < j \le N} \mathcal{L}_{\mathrm{Miwa}}([u], \lambda_i, \lambda_j)\, \eta_i \wedge \eta_j.$$

Continuum limit of a Lagrangian 2-form

$$L(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2)$$

Suitable choice $\Rightarrow$ leading order cancellation

$\big\downarrow$ Miwa shifts, Taylor expansion

$$\mathcal{L}_{\mathrm{disc}}([u], \lambda_1, \lambda_2)$$

$\big\downarrow$ Euler-Maclaurin formula

$$\mathcal{L}_{\mathrm{Miwa}}([u], \lambda_1, \lambda_2) \quad = \quad \sum_{i,j=1}^{\infty} (-1)^{i+j} 4 \frac{\lambda_1^i}{i} \frac{\lambda_2^j}{j} \mathcal{L}_{i,j}[u]$$

$\big\downarrow$ $\qquad\qquad$ $\big\downarrow$

$$\sum_{1 \leq i < j \leq N} \mathcal{L}_{\mathrm{Miwa}}([u], \lambda_i, \lambda_j)\, \eta_i \wedge \eta_j \qquad = \qquad \sum_{1 \leq i < j \leq N} \mathcal{L}_{i,j}[u] \,\mathrm{d}t_i \wedge \mathrm{d}t_j$$

Continuum limit of the Lagrangian for H1

Lagrangian for (lpKdV)

$$L(\square) = \frac{1}{2}\left(U - U_{i,j} - \lambda_i^{-1} - \lambda_j^{-1}\right)\left(U_i - U_j + \lambda_i^{-1} - \lambda_j^{-1}\right) + \left(\lambda_i^{-2} - \lambda_j^{-2}\right)\log\left(1 + \frac{U_i - U_j}{\lambda_i^{-1} - \lambda_j^{-1}}\right).$$

A well-chosen representative among many equivalent Lagrangians.

Continuum limit procedure:

- Miwa correspondence:

$$U = U(\mathbf{n}) = u(t_1, t_2, \ldots, t_n),$$
$$U_i = U(\mathbf{n} + \mathfrak{e}_i) = u\left(t_1 - 2\lambda_i, t_2 + 2\frac{\lambda_i^2}{2}, \ldots, t_n + 2(-1)^n \frac{\lambda_i^N}{N}\right)$$

- Series expansion
- Euler-Maclaurin formula

Coefficients (after some post-limit simplifications)

$$\mathcal{L}_{1,2} = \tfrac{1}{2} u_1 u_2 \qquad \mathcal{L}_{1,3} = -u_1^3 + \tfrac{1}{2} u_{11}^2 + \tfrac{1}{2} u_1 u_3$$

$$\mathcal{L}_{1,4} = \tfrac{1}{2} u_1 u_4 \qquad \mathcal{L}_{1,5} = -\tfrac{5}{2} u_1^4 + 5 u_1 u_{11}^2 - \tfrac{1}{2} u_{111}^2 + \tfrac{1}{2} u_1 u_5$$

$$\mathcal{L}_{2,3} = -3u_1^2 u_2 + u_{11} u_{12} - u_{111} u_2 + \tfrac{1}{2} u_2 u_3$$

$$\mathcal{L}_{2,4} = \tfrac{1}{2} u_2 u_4$$

$$\begin{aligned}\mathcal{L}_{2,5} = {} & -10 u_1^3 u_2 + 10 u_1 u_{11} u_{12} - 5 u_{11}^2 u_2 - 10 u_1 u_{111} u_2 - u_{111} u_{112} + \\ & u_{1111} u_{12} - u_{11111} u_2 + \tfrac{1}{2} u_2 u_5\end{aligned}$$

$$\mathcal{L}_{3,4} = -u_{11} u_{14} + \tfrac{1}{2} u_3 u_4$$

$$\begin{aligned}\mathcal{L}_{3,5} = {} & 18 u_1^5 + 30 u_1^3 u_{111} - 10 u_1^3 u_3 + 6 u_{11}^2 u_{111} + 8 u_1 u_{111}^2 - 6 u_1 u_{11} u_{1111} + \\ & 3 u_1^2 u_{11111} + 10 u_1 u_{11} u_{13} - 5 u_{11}^2 u_3 - 10 u_1 u_{111} u_3 - \tfrac{1}{2} u_{1111}^2 + \\ & u_{111} u_{11111} - u_{111} u_{113} + u_{1111} u_{13} - u_{11} u_{15} - u_{11111} u_3 + \tfrac{1}{2} u_3 u_5\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{4,5} = {} & -10 u_1^3 u_4 + 10 u_1 u_{11} u_{14} - 5 u_{11}^2 u_4 - 10 u_1 u_{111} u_4 - u_{111} u_{114} + \\ & u_{1111} u_{14} - u_{11111} u_4 + \tfrac{1}{2} u_4 u_5\end{aligned}$$

Coefficients (after some post-limit simplifications)

$\mathcal{L}_{1,2} = \frac{1}{2} u_1 u_2$

$\mathcal{L}_{1,3} = -u_1^3 + \frac{1}{2} u_{11}^2 + \frac{1}{2} u_1 u_3$

$\mathcal{L}_{1,4} = \frac{1}{2} u_1 u_4$

$\mathcal{L}_{1,5} = -\frac{5}{2} u_1^4 + 5 u_1 u_{11}^2 - \frac{1}{2} u_{111}^2 + \frac{1}{2} u_1 u_5$

$\mathcal{L}_{2,3} = -3u_1^2 u_2 + u_{11} u_{12} - u_{111} u_2 + \frac{1}{2} u_2 u_3$

$\mathcal{L}_{2,4} = \frac{1}{2} u_2 u_4$

$$\mathcal{L}_{2,5} = -10u_1^3 u_2 + 10 u_1 u_{11} u_{12} - 5u_{11}^2 u_2 - 10 u_1 u_{111} u_2 - u_{111} u_{112} + u_{1111} u_{12} - u_{11111} u_2 + \frac{1}{2} u_2 u_5$$

$\mathcal{L}_{3,4} = -u_{11} u_{14} + \frac{1}{2} u_3 u_4$

$$\mathcal{L}_{3,5} = 18u_1^5 + 30u_1^3 u_{111} - 10u_1^3 u_3 + 6u_{11}^2 u_{111} + 8u_1 u_{111}^2 - 6u_1 u_{11} u_{1111} + 3u_1^2 u_{11111} + 10 u_1 u_{11} u_{13} - 5u_{11}^2 u_3 - 10 u_1 u_{111} u_3 - \frac{1}{2} u_{1111}^2 + u_{111} u_{11111} - u_{111} u_{113} + u_{1111} u_{13} - u_{11} u_{15} - u_{11111} u_3 + \frac{1}{2} u_3 u_5$$

$$\mathcal{L}_{4,5} = -10u_1^3 u_4 + 10 u_1 u_{11} u_{14} - 5u_{11}^2 u_4 - 10 u_1 u_{111} u_4 - u_{111} u_{114} + u_{1111} u_{14} - u_{11111} u_4 + \frac{1}{2} u_4 u_5$$

Continuum limits of ABS equations

$$Q1_{\delta=0} \quad \to \quad v_3 = v_{111} - \frac{3v_{11}^2}{2v_1} \qquad \text{Schwarzian KdV}$$

$$Q1_{\delta=1} \quad \to \quad v_3 = v_{111} - \frac{3}{2}\frac{v_{11}^2 - \frac{1}{4}}{v_1}$$

$$Q2 \quad \to \quad v_3 = v_{111} - \frac{3}{2}\frac{v_{11}^2 - \frac{1}{4}}{v_1} - \frac{3}{2}\frac{v_1^3}{v^2}$$

$$Q3_{\delta=0} \quad \to \quad v_3 = v_{111} - \frac{3}{2}\frac{v_{11}^2 - \frac{1}{4}}{v_1} + \frac{1}{2}v_1^3$$

$$Q3_{\delta=1} \quad \to \quad v_3 = v_{111} - \frac{3}{2}\frac{v_{11}^2 - \frac{1}{4}}{v_1} + \frac{1}{2}v_1^3 - \frac{3}{2}\frac{v_1^3}{\sin(v)^2}$$

$$Q4 \quad \to \quad v_3 = v_{111} - \frac{3}{2}\frac{v_{11} - \frac{1}{4}}{v_1} - \frac{3}{2}\wp(2v)v_1^3 \qquad \text{Krichever-Novikov}$$

$$H1 \quad \to \quad v_3 = v_{111} + 3v_1^2 \qquad \text{Potential KdV}$$

$$H3_{\delta=0} \quad \to \quad v_3 = v_{111} + \frac{1}{2}v_1^3 \qquad \text{Potential mKdV}$$

All with their hierarchies

Conclusions

- The pluri-Lagrangian (or Lagrangian multiform) principle is a widely applicable characterization of integrability:

  It applies to integrable ODEs and PDEs, and to integrable difference equations of any dimension.
- (Almost-)closedness of the pluri-Lagrangian form, i.e. $\mathrm{d}\mathcal{L} = \mathit{const}$ is related to variational symmetries.
- Tools to construct pluri-Lagrangian structures:
  - Variational symmetries
  - Continuum limits
- Open questions:
  - Pluri-Lagrangian 3-form systems
  - Precise relation to (bi-)Hamiltonian structures
  - ...

Selected references

General	▶ Lobb, Nijhoff. Lagrangian multiforms and multidimensional consistency. J. Phys. A. 2009. ▶ Suris. Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms. J. Geometric Mechanics, 2013 ▶ Boll, Petrera, Suris. What is integrability of discrete variational systems? Proc. R. Soc. A. 2014. ▶ Suris, V. On the Lagrangian structure of integrable hierarchies. In: Advances in Discrete Differential Geometry, Springer. 2016.
Variational symmetries	▶ Petrera, Suris. Variational symmetries and pluri-Lagrangian systems in classical mechanics. J. Nonlin. Math. Phys., 2017. ▶ Petrera, V. Variational symmetries and pluri-Lagrangian hierarchies. In preparation.
Continuum limits	▶ V. Continuum limits of pluri-Lagrangian systems. Journal of Integrable Systems, 2019 ▶ V. A variational perspective on continuum limits of ABS and lattice GD equations, arXiv:1811.01855

Thank you for your attention!