The Multisymplectic Form Formula for Lagrangian PDEs

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Lagrangian Mechanics

Setup:

- Configuration space Q – coordinates $(q^1, \dots, q^N)$.
- Phase space TQ – coordinates $(q^1, \dots, q^N, \dot{q}^1, \dots, \dot{q}^N)$.
- Lagrange function $L : TQ \to \mathbb{R}$.
- Solutions curves $q : [a, b] \to Q : t \mapsto (q^1(t), \dots, q^N(t))$ are critical points of the action

$$
\begin{aligned}
S(\alpha) &= \int_a^b L\left(q^1(t), \dots, q^N(t), \dot{q}^1(t), \dots, \dot{q}^N(t)\right) \mathrm{d}t \\
&= \int_a^b L(q(t), \dot{q}(t)) \mathrm{d}t.
\end{aligned}
$$

If $L =$ (kinetic energy) $-$ (potential energy), this describes Newtonian mechanics.

The Euler-Lagrange Equations

Fix endpoints $q(a), q(b)$, take any variation δq of $q(t)$:

$$\begin{aligned} 0 = \delta S &= \int_a^b \delta L(q(t), \dot{q}(t)) \, \mathrm{d}t \\ &= \int_a^b \frac{\partial L}{\partial q^i} \, \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \, \delta \dot{q}^i \, \mathrm{d}t \\ &= \int_a^b \left(\frac{\partial L}{\partial q^i} \, \delta q^i - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}^i} \right) \delta q^i \right) \mathrm{d}t \quad + \frac{\partial L}{\partial \dot{q}^i} \, \delta q^i \bigg|_{t=a}^{t=b} \\ &= \int_a^b \left(\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}^i} \right) \right) \delta q^i \, \mathrm{d}t. \end{aligned}$$

The Euler-Lagrange equations follow:

$$\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}^i} = 0.$$

The Symplectic Form

Leave endpoints $q(a), q(b)$ free, restricts to curves $q(t)$ which solve the EL-equations:

$$\begin{aligned}
\delta S &= \int_a^b \frac{\partial L}{\partial q^i}\ \delta q^i + \frac{\partial L}{\partial \dot{q}^i}\ \delta \dot{q}^i \,\mathrm{d}t \\
&= \int_a^b \left(\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} \right) \delta q^i \mathrm{d}t \quad + \frac{\partial L}{\partial \dot{q}^i}\ \delta q^i \,\bigg|_{t=a}^{t=b} \\
&= \frac{\partial L}{\partial \dot{q}^i}\ \delta q^i \,\bigg|_{t=a}^{t=b}.
\end{aligned}$$

Introduce conjugate momenta $p_i := \dfrac{\partial L}{\partial \dot{q}^i}$, then

$$\delta S = p_i\ \delta q^i \,\bigg|_{t=a}^{t=b}.$$

The Symplectic Form

In terms of differentials:

$$\delta S = \mathrm{d}S(\delta q, \delta \dot{q}) \qquad \text{and} \qquad \delta q^i = \mathrm{d}q^i(\delta q, \delta \dot{q}),$$

so

$$\mathrm{d}S = p_i \; \mathrm{d}q^i \Big|_{t=a}^{t=b}.$$

Take exterior derivative:

$$0 = \mathrm{d}p_i \wedge \mathrm{d}q^i \Big|_{t=a}^{t=b} = \mathrm{d}p_i(b) \wedge \mathrm{d}q^i(b) - \mathrm{d}p_i(a) \wedge \mathrm{d}q^i(a).$$

The symplectic two-from $\omega := \mathrm{d}p_i \wedge \mathrm{d}q^i$ is a conserved quantity.

Lagrangian Field Theory

Setup:

- Spacetime manifold X – coordinates $(x^\mu) = (x^0, \dots, x^n)$
- Configuration space: trivial vector bundle $Y = X \times F$ – coordinates $(x^\mu, y^i) = (x^1, \dots, x^n, y^1, \dots, y^m)$.
- Phase space $J^1 Y$ – coordinates (x^μ, y^i, y^i_μ).

  Section $\phi : X \to Y$ extends to

$$j^1\phi : X \to J^1 Y : x \mapsto (x^\mu, \phi(x)^i, \partial_\mu \phi^i(x)).$$

Example. Mechanics:

$$\begin{aligned} X = \mathbb{R} \quad &- \quad \text{time} \\ F = Q \quad &- \quad \text{configuration space} \\ J^1 Y = X \times TQ \quad &- \quad \text{time} \times \text{phase space} \end{aligned}$$

Euler Lagrange Equations

Setup:

- Lagrange function:

$$\mathcal{L} : J^1 Y \to \Omega^{n+1} X : (x^\mu, y^i, y^i_\mu) \mapsto L(y^i, y^i_\mu)\, \mathrm{d}x_0 \wedge \ldots \wedge \mathrm{d}x_n.$$

- Action:

$$S(\phi) = \int_{U \subset X} L(\phi^i, \partial_\mu \phi^i)\, \mathrm{d}x_0 \wedge \ldots \wedge \mathrm{d}x_n.$$

Form and derivation of Euler-Lagrange equations

$$\frac{\partial L}{\partial y^i} - \frac{\mathrm{d}}{\mathrm{d}x^\mu} \frac{\partial L}{\partial y^i_\mu} = 0$$

are very similar to mechanics.

The Multisymplectic Form Formula – MPS version

Considering the boundary terms leads to Multisymplectic form formula:

$$\int_{\partial U} \iota_{\text{something}}\, \iota_{\text{something similar}}\, (j^1\phi)^* \omega = 0,$$

where

- ι denotes the interior product and * the pull-back.
- $\phi : U \to J^1 Y$ is any solution of the EL equations.
- ω is the Multisymplectic $(n+2)$-form:

$$\omega = \mathrm{d} \left(\frac{\partial L}{\partial y^i_\mu} \mathrm{d} y^i \wedge \mathrm{d}^n x_\mu + \left(L - \frac{\partial L}{\partial y^i_\mu} y^i_\mu \right) \mathrm{d}^{n+1} x \right),$$

where $\quad \mathrm{d}^{n+1} x = \mathrm{d} x^0 \wedge \ldots \mathrm{d} x^n,$

and $\quad \mathrm{d}^n x_\mu = (-1)^\mu \, \mathrm{d} x^0 \wedge \ldots \wedge \widehat{\mathrm{d} x^\mu} \wedge \ldots \wedge \mathrm{d} x^n.$

- "something" is a first variation of ϕ (extended to $J^1 Y$).

Horizontal and Vertical One-Forms

- Phase space $J^\infty Y$ – coordinates $(x^\mu, y^i, y^i_\mu, y^i_{\mu\nu}, \ldots, y^i_{\mathcal{I}}, \ldots)$.

  Section $\phi : X \to Y$ extends to

$$j^\infty\phi : X \to J^\infty Y : x \mapsto (x^\mu, \phi(x)^i, \partial_\mu \phi^i(x), \partial_{\mu\nu}\phi^i(x), \ldots).$$

- Horizontal one-forms

$$\mathscr{H} = \mathrm{span}\{\mathrm{d}x^0, \ldots, \mathrm{d}x^n\}$$

  annihilate vertical vector fields.

- Vertical one-forms

$$\mathscr{V} = \mathrm{span}\{\mathrm{d}y^i - y^i_\mu \mathrm{d}x^\mu, \mathrm{d}y^i_\nu - y^i_{\nu\mu}\mathrm{d}x^\mu, \ldots, y^i_{\mathcal{I}} - y^i_{\mathcal{I}\mu}\mathrm{d}x^\mu, \ldots\}$$

  annihilate extended vector fields.

Type (h,v)-Forms

- Type (h,v)-forms on $J^\infty Y$

$$\Omega^{h,v}(J^\infty Y) = \operatorname{span}\{\alpha^1 \wedge \ldots \wedge \alpha^h \wedge \beta^1 \wedge \ldots \wedge \beta^v \mid \alpha^j \in \mathscr{H}, \beta^j \in \mathscr{V}\}$$
$$\subset \Omega^{h+v}(J^\infty Y)$$

- Exterior derivative $\mathrm{d} : \Omega^k \to \Omega^{k+1}$ splits uniquely into

$$\mathrm{d}_{\mathscr{H}} : \Omega^{h,v} \to \Omega^{h+1,v} \qquad \text{and} \qquad \mathrm{d}_{\mathscr{V}} : \Omega^{h,v} \to \Omega^{h,v+1}$$

such that

$$\mathrm{d} = \mathrm{d}_{\mathscr{H}} + \mathrm{d}_{\mathscr{V}}.$$

The Variational Bicomplex

The variational bicomplex on $J^\infty Y$:

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & & & \vdots & & \vdots & & \\
\uparrow \mathrm{d}_{\mathscr{V}} & & \uparrow \mathrm{d}_{\mathscr{V}} & & & & \uparrow \mathrm{d}_{\mathscr{V}} & & \uparrow \mathrm{d}_{\mathscr{V}} & & \\
\Omega^{0,2} & \xrightarrow{\mathrm{d}_{\mathscr{H}}} & \Omega^{1,2} & \xrightarrow{\mathrm{d}_{\mathscr{H}}} & \dots & \xrightarrow{\mathrm{d}_{\mathscr{H}}} & \Omega^{n,2} & \xrightarrow{\mathrm{d}_{\mathscr{H}}} & \Omega^{n+1,2} & \xrightarrow{\mathrm{d}_{\mathscr{H}}} & 0 \\
\uparrow \mathrm{d}_{\mathscr{V}} & & \uparrow \mathrm{d}_{\mathscr{V}} & & & & \uparrow \mathrm{d}_{\mathscr{V}} & & \uparrow \mathrm{d}_{\mathscr{V}} & & \\
\Omega^{0,1} & \xrightarrow{\mathrm{d}_{\mathscr{H}}} & \Omega^{1,1} & \xrightarrow{\mathrm{d}_{\mathscr{H}}} & \dots & \xrightarrow{\mathrm{d}_{\mathscr{H}}} & \Omega^{n,1} & \xrightarrow{\mathrm{d}_{\mathscr{H}}} & \Omega^{n+1,1} & \xrightarrow{\mathrm{d}_{\mathscr{H}}} & 0 \\
\uparrow \mathrm{d}_{\mathscr{V}} & & \uparrow \mathrm{d}_{\mathscr{V}} & & & & \uparrow \mathrm{d}_{\mathscr{V}} & & \uparrow \mathrm{d}_{\mathscr{V}} & & \\
\Omega^{0,0} & \xrightarrow{\mathrm{d}_{\mathscr{H}}} & \Omega^{1,0} & \xrightarrow{\mathrm{d}_{\mathscr{H}}} & \dots & \xrightarrow{\mathrm{d}_{\mathscr{H}}} & \Omega^{n,0} & \xrightarrow{\mathrm{d}_{\mathscr{H}}} & \Omega^{n+1,0} & \xrightarrow{\mathrm{d}_{\mathscr{H}}} & 0
\end{array}$$

A “vertical” extension of the de Rham complex on X:

$$\Omega^0(X) \xrightarrow{\mathrm{d}_X} \Omega^1(X) \xrightarrow{\mathrm{d}_X} \dots \xrightarrow{\mathrm{d}_X} \Omega^n(X) \xrightarrow{\mathrm{d}_X} \Omega^{n+1}(X) \xrightarrow{\mathrm{d}_X} 0$$

Integration of Type (h,v)-Forms

Setup:

- $\alpha \wedge \beta \in \Omega^{h,v}(J^\infty Y)$, with
$$\alpha \in \Omega^{h,0}(J^\infty Y) = \Omega^h(X) \qquad \text{and} \qquad \beta \in \Omega^{0,v}(J^\infty Y)$$
- M – an h-dimensional submanifold of the base manifold X.
- $J^\infty Y$ is locally trivial: $J^\infty Y = X \times F^\infty Y$.
- $\Lambda^v(F^\infty Y)$ – the space of anti-symmetric v-tensors on $F^\infty Y$.
- An (h, v)-form can be seen as a $\Lambda^v(F^\infty Y)$-valued h-form on X.

The integral of an (h, v)-form on M is an anti-symmetric v-tensor on $F^\infty Y$:
$$\left(\int_M \alpha \wedge \beta\right)(W_1, \ldots, W_v) = \int_M \underbrace{\beta(W_1, \ldots, W_v)}_{\in\mathbb{R}} \underbrace{\alpha}_{\in\Omega^{h,0}}.$$

The Multisymplectic Form Formula – Take 2

- Volume element $\mathrm{d}^n V$ on ∂U.
- n_μ coordinates of the unit normal vector to ∂U.
- Normal momenta:

$$\pi_i = \frac{\partial L}{\partial y^i_\mu} n_\mu .$$

$\Big($Compare conjugate momenta in mechanics: $p^i = \frac{\partial L}{\partial \dot{q}^i}$.$\Big)$

Considering the boundary terms yields

Theorem (Multisymplectic Form Formula)

$$\int_{\partial U} \mathrm{d}_{\mathscr{V}} \pi_i \wedge \mathrm{d}_{\mathscr{V}} y^i \wedge \mathrm{d}^n V = 0.$$

The Multisymplectic Form Formula – Take 2

Proof. Along solutions of the Euler-Lagrange equations we have:

$$\begin{aligned}
\delta S &= \int_U \left(\frac{\partial L}{\partial y^i} \delta y^i + \frac{\mathrm{d}}{\mathrm{d}x^\mu} \frac{\partial L}{\partial y^i_\mu} \delta y^i_\mu \right) \mathrm{d}^{n+1} x \\
&= \int_U \left(\frac{\partial L}{\partial y^i} - \frac{\mathrm{d}}{\mathrm{d}x^\mu} \frac{\partial L}{\partial y^i_\mu} \right) \delta y^i \mathrm{d}^{n+1} x + \int_{\partial U} \left(\frac{\partial L}{\partial y^i_\mu} \delta y^i \right) \mathrm{d}^n x_\mu \\
&= \int_{\partial U} \left(\frac{\partial L}{\partial y^i_\mu} \delta y^i \right) \mathrm{d}^n x_\mu \\
\Rightarrow \mathrm{d} S &= \int_{\partial U} \frac{\partial L}{\partial y^i_\mu} \mathrm{d}_{\mathscr{V}} y^i \wedge \mathrm{d}^n x_\mu = \int_{\partial U} \pi_i \mathrm{d}_{\mathscr{V}} y^i \wedge \mathrm{d}^n V.
\end{aligned}$$

Take the exterior derivative again:

$$0 = \int_{\partial U} \mathrm{d}_{\mathscr{V}} \pi_i \wedge \mathrm{d}_{\mathscr{V}} y^i \wedge \mathrm{d}^n V.$$

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